*Subscription, $1.50 Per Year in Advance* *Single Copies, 20c*

VOL. XIII UNIVERSITY, LA., OCTOBER, 1938 No. 1

Application for second-class entry at University, Louisiana, is pending.

Published monthly excepting June, July, August, September, by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

☞All Business Communications should be addressed to the Editor and Manager.



*This Journal is dedicated to the following aims:*

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

# President Hotchkiss on the Place of Mathematics in Our Schools

I have long been distressed by the attitude of many of the secondary schools in the policy of eliminating mathematics as a required subject. Mathematics and the other sciences are the only subjects in the curricula of our schools which permit of rigorous, logical thinking, and I do not believe that any graduate of our schools can be considered to be properly educated until he has been introduced to this quality of thinking. I would like to differentiate between the *scientific* discipline of thinking and the *letters and arts* discipline as exhibiting the difference that is found between quantitative and qualitative analysis in chemistry. We are the victims, at present, of too many "qualitative" thinkers, and we lack, too seriously, the benefit of the thoughts of "quantitative" thinkers.

I think this tendency on the part of school administrators to eliminate mathematics is a part of the grievous tendency on the part of teachers to emphasize the difficulties of the subject rather than the simplicities. Too many teachers feed their egotism by impressing the students with the difficulty of the subject and the consequent smartness of the teacher who understands it.

Mathematics is a fundamental tool science, and every person should find it an aid to more intelligent thinking. The better his training in mathematics, the more clearly is the average citizen going to be able to think and arrive at conclusions that are valid.

WM. O. HOTCHKISS,
*Rensselær Polytechnic Institute.*

# On Harmonic Separation

By ARCHIBALD HENDERSON and JOHN WAYNE LASLEY, JR.
*University of North Carolina*

## PART I—HISTORICAL

*Introduction.* The problem of the harmonic separation of four points on a line is basic for projective geometry. Its history is interesting and too little known. The number of methods for determining the harmonic conjugate of a point as to two points collinear with it is surprisingly large. In the historical development of the concept the linear construction seems to have played the predominant role. This paper purports to give an account of this development and to point out other constructions, mostly quadratic, which also will give the same well-known result: harmonic separation.

*The Pythagoreans.* So far as is known* the discoverer of the invariance of the harmonic relation is unknown. We are not any too sure of the reason for the name "harmonic". The very name suggests the harmonies of music. Certain it is that $C$, $G$, and $C'$ of the diatonic scale sound in harmony. Their frequencies are: 256, 384 and 512 in the order named. These numbers are in the ratio of 1, 3/2 and 2, which are in arithmetical progression. Their reciprocals 1, 2/3 and 1/2 are in harmonical progression, so-called, and are proportional to the lengths of the strings vibrating with these frequencies and producing the pleasing sounds above mentioned. This type of origin of the term "harmonic" is attributed to the Pythagoreans.

*Archytas.* Durell† says that the word harmonic was in use at the beginning of the fourth century B. C., although probably not in the sense used in this paper. There is no definite evidence linking any of the Pythagoreans with the cross-ratio or its invariance. Archytas ($-400$) seems more likely than any other to have possessed this knowledge. He, according to Allman,* may have been acquainted with projection, and possibly used it in his solution of the problem of the duplication of the cube, which he accomplished by the use of two mean proportionals.

*Coolidge, J. L., *American Mathematical Monthly*, Vol. 41 (1934), p. 217.
†Durell, C. V., *Geometry for Advanced Students*, Vol. 1 (1909), p. 93.
‡Allman, G. J., *Greek Geometry from Thales to Euclid* (1889), p. 122 et seq.

*Aristotle.* There may be found in Aristotle $(-340)$, according to Heiberg,* a construction which has the potentialities of harmonic determination.

*Euclid.* Chasles† regards it as probable that Euclid $(-300)$ knew of the cross-ratio and of the fact that it was invariant under projection. This view is shared by Coolidge and Durell.

*Apollonius.* Apollonius is accredited by Heiberg with the theorem that, if from two given points straight lines be drawn to another point in such a way that their lengths are in a constant ratio, the locus of this latter point is a circle. We have here the Circle of Apollonius about two points, and we have with it all the potentialities of harmonic separation.

*Serenus.* Ball‡ mentions Serenus of Antissa, gives him the very problematical date of c. 70, and says that he "laid down a fundamental proposition on transversals." Cajori§ speaks of him as one who gave a theorem on which is based "the modern theory of harmonics." This theorem, because of its historic interest, is quoted here (Fig. 1): "If from

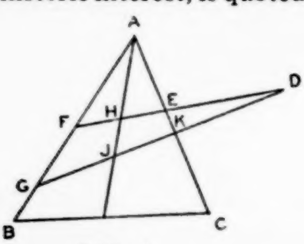



FIG. 1

$D$ we draw $DF$ cutting triangle $ABC$ [$AC$ in $E$, and $AB$ in $F$], and choose $H$ so that $DE/DF = EH/HF$, and draw $AH$, then every transversal through $D$, such as $DG$, will be divided by $AH$ so that $DK/DG = KJ/JG$." No clue is given as to how $H$ may be located so as to meet the requirement of the proposition. It seems tacitly assumed that such a construction was well known at that time. The emphasis is rather on the persistence in $J$ of the property enjoyed by $H$. Since $H$ is none other than the harmonic conjugate of $D$ as to $E$ and $F$, this strongly suggests that the concept of four harmonic points was a commonplace in the time of Serenus.

*Menelaus.* It does not seem so evident, however, that this was known in the time of Menelaus, about 100 A. D. Smith‡‡ says that

*Heiberg, J. L., *Apollonius* (Edition of his works), Vol. 2, pp. 180-184.

†Chasles, M., *Aperçu Historique sur l'origine et les developpement des Méthodes en Géometrie* (1875), 2 ed., Paris.

‡Ball, W. W. R., *History of Mathematics*, p. 94.

§Cajori, F., *History of Mathematics*, p. 46.

‡‡Smith, D. E., *History of Mathematics*, Vol. 1, p. 127.

Menelaus knew the anharmonic property of four points and its invariance under projection, although these are usually attributed to Pappus, two hundred years later. It is thought by Heath[*] that Menelaus in proving a certain theorem regarding great circle arc-lengths assumed as known the invariance of the cross-ratio of four great circles drawn from a point on a sphere in relation to any great circle intersecting them all; viz., that if $A,B,C,D$; $A',B',C',D'$ be the points where two transversals cut the arcs of the great circles from $O'$ (Fig. 2), then

$$\frac{\operatorname{Sin} AD}{\operatorname{Sin} DC}\cdot\frac{\operatorname{Sin} BC}{\operatorname{Sin} AB}=\frac{\operatorname{Sin} A'D'}{\operatorname{Sin} D'C'}\cdot\frac{\operatorname{Sin} B'C'}{\operatorname{Sin} A'D'}$$

Coolidge[†] is not so optimistic about Menelaus and informs us that the most recent evidence indicates the absence of both the plane theorem and the spherical theorem from the writings of Menelaus. However, from studies by Björnbo, he infers that Menelaus was cognizant of the theorem in spherical geometry.

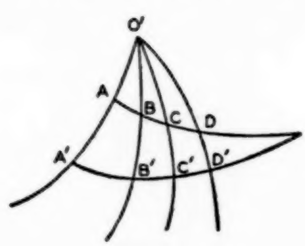



FIG. 2

*Pappus.* Coolidge cites as the earliest work in which the plane theorem is to be found the work of Pappus, who wrote perhaps two hundred years after Menelaus. Copies of Pappus[‡] have come down to us and give us a clear picture of his construction for the harmonic conjugate. Let us in the more modern sense define the ratio $(ABC)$ of three points on a line $l$ to be that of the line segment $AC$ to the line segment $BC$, thus

$$(1)\qquad (ABC)=AC/BC$$

and speak of this as the ratio in which the point $C$ divides the pair of points $A$ and $B$ on the line $l$. Let us call $D$ the harmonic conjugate of $C$ as to $A$ and $B$ if $D$ divides $A$ and $B$ externally in the same ratio

[*]Heath, T. L., *Greek Mathematics*, Vol. 2, p. 268.
[†]Coolidge, J. L., *loc. cit.*, p. 218.
[‡]Pappus, *Mathematicae Collectiones* (Edition by Federicus Commandinus), Bononiae (1659).

as that in which $C$ divides $A$ and $B$ internally. In the light of this meaning we have for $D$ the relation

$$AC/BC = -AD/BD \tag{2}$$

where the negative sign has been employed on the right hand side of the equation to account for the directions of the line segments. With this understanding, Pappus' construction for $D$ if $A, B$ and $C$ are known is equivalent to the following (Fig. 3): At $C$ erect a perpendicular

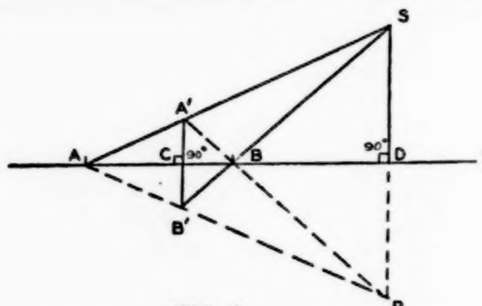



FIG. 3

to $l$ and on it lay off $CA'$ and $CB'$ equal, of any arbitrary length, and in opposite directions. Draw $AA'$ and $BB'$ meeting in $S$. Drop a perpendicular from $S$ to $l$, meeting it in the required point $D$. For, by similar triangles, $AC/AD = CA'/DS$. Also, by similar triangles, $BC/BD = CB'/DS$. But $CB' = -CA'$. Consequently, upon dividing one equation by the other, and clearing of fractions we have (2), showing that $D$ is the harmonic conjugate of $C$ as to $A$ and $B$ upon $l$.

*Desargues.* Here the matter rested* for thirteen centuries. Not until the time of Desargues (c. 1640) was the notion extended. Desargues was first in modern times to employ the ratio of two ratios, called by Clifford the cross-ratio. He gave us the theory of involution as we know it today, exemplifying it beautifully in his celebrated theorem on conics. We are indebted to him also for his theorem on triangles, perhaps even more celebrated. He used cross-ratio as the basis for the first systematic development of what we now designate as projective geometry.

*Pascal.* There seems to be no doubt† that Pascal (c. 1650) knew of the invariance of the cross ratio under projective transformation.

*La Hire.* It appears certain also‡ that La Hire (c. 1690) must have known of Desargues' work, but nowhere does he make mention

*Coolidge, J. L., *loc. cit.*, p. 218.
†Coolidge, J. L., *loc. cit.*, p. 218.
‡Coolidge, J. L., *loc. cit.*, p. 219.

of it. Coolidge claims that La Hire added little to projective geometry. This view of the matter is not shared by Wilkinson* who regards La Hire's *Sectiones Conicae* of 1685 as the link between the ancient and the modern geometries. This work is exceedingly rare. Wilkinson gives a short summary and translation of the introductory portion of the work. Chasles† says that this work of La Hire is an "introduction and a development in the sequel of the easy and general demonstrations of the theorems which have roused in the ancients long and painful development. This is wherein consists the novelty and merit of La Hire's work." In La Hire's work is found the equivalent of the following construction for the harmonic conjugate (Fig. 4): On $C$ at any conven-

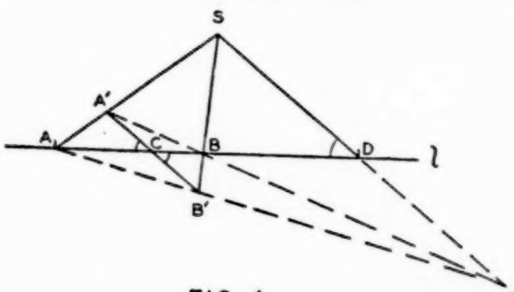



FIG 4

ient angle draw any line other than $l$ and on this other line lay off $CA'$ and $CB'$ equal and in opposite directions. Draw $AA'$ and $BB'$ meeting in $S$. On $S$ draw $SD$ parallel to $A'B'$, meeting $l$ in the required point $D$. The proof is precisely as before, depending, as there, on the similarity of triangles. The forward step made here by La Hire is the replacement of the perpendiculars of Pappus by lines such as $A'B'$ at an arbitrary angle. La Hire‡ went further and freed the construction of all metric features. Due to him is the construction for $D$ purely projective in character now almost universally used. An account of this construction is to be found elsewhere in this paper.

*Maclaurin.* Wilkinson makes the statement that Maclaurin (c. 1740) had "used La Hire's investigations in the appendix to his 'Algebra'."

*Carnot.* Smith§ says that Carnot (c. 1803) for the first time gave a general form to the theory of projection in his *Géometrie de Position.* Carnot first introduced negative magnitudes into projective geometry. Thus the cross-ratio took on algebraic signs and the direction of a line

*Wilkinson, T. T., *Ladies' and Gentleman's Diary*, Vol. 17-21 (1861), p. 91.
†Chasles, M., *loc. cit.*, p. 123.
‡De la Hire, P., *Sectiones Conicae* (1685), book 1, proposition 20.
§Smith, D. E., *loc. cit.*, p. 333.

segment was cared for in the analysis. It appears, strangely enough, that* the invariance of the cross-ratio escaped the attention of Poncelet (c. 1830), although he "contributed the ideas of central projection and homology to projective geometry."

*Steiner*. When Steiner (c. 1840) appeared geometry took on new life. His contributions to geometry were many and varied. A noteworthy one involving cross-ratio is his theorem that a variable tangent to a conic meets four fixed tangents of the conic in points with a constant cross-ratio. This is the dual of a theorem of Chasles. Steiner gave us his well-known construction for a projective correspondence, for an involution and for the double points of each. Steiner, together with Chasles, according to Cajori,† "raised synthetic geometry to an honored and respected position by the side of analysis."

*Chasles*. Chasles (c. 1845) brought back to notice the work of Desargues, which was submerged under the work of Descartes. It seems that Chasles took the next step in generalizing the construction of the harmonic conjugate. If $D$ is the harmonic conjugate of $C$ as to $A$ and $B$, we have from (2) the relation

$$\frac{AC}{BC}\bigg/\frac{AD}{BD}=-1 \tag{3}$$

which we shall write more briefly as

$$(ABCD)=-1 \tag{4}$$

If now $(ABCD)$ instead of being equal to $-1$ is equal to some other real number, say $\lambda$, we have the problem: Given three points $A, B$ and $C$ of a line $l$ and a real number $\lambda$, to construct a point $D$ on $l$ such that $(ABCD)=\lambda$. Chasles‡ in his *Géometrie Supérieure* solved this problem. His method is essentially as follows: On $C$ take a line other than $l$ and on it lay off any arbitrary line segment $CB'$. With $CB'$ as a unit lay off $CA'=\lambda\cdot CB'$. The point $A'$ will fall on the same side of

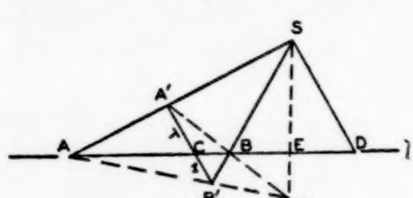



FIG. 5

*Coolidge, J. L., *loc. cit.*, p. 220.
†Cajori, F., *loc. cit.*, p. 294.
‡Chasles, M., *Géometrie Supérieure*, p. 10.

$C$ as $B'$, if $\lambda$ positive; on the opposite side of $C$ from $B'$, if (as in Fig. 5) $\lambda$ is negative. Now draw $AA'$ and $BB'$ meeting in $S$. Then $SD$, drawn parallel to $A'B'$, will meet $l$ in the required point $D$. For, by similar triangles $AC/AD = CA'/DS$. Also from similar triangles we have $BC/BD = CB'/DS$. Dividing the first of these equations by the second we obtain $(ABCD) = CA'/CB' = \lambda$. Since $D$ is unique, and this construction obtains one such point for us, the problem is solved. Chasles* gave us the theory of homography and reciprocity. He discovered the theorem that four fixed points on a conic join to a variable fifth point on the conic in lines of a constant cross-ratio.

*Von Staudt.* Von Staudt (c. 1850) freed the cross-ratio from metric bondage by means of his theory of the "throw" (Wurf). He brought the imaginary into its own in projective geometry. He used the elliptic involution to define the imaginary. He approached the conic through the theory of correlations, placing the imaginary conic on a parity with the real. Coolidge† places Von Staudt above Steiner, a most desirable eminence for a geometer.

*Davies.* Davies‡ (c. 1850) gave a construction for the harmonic conjugate. It is the same as that of La Hire, and need not be given here. Davies employs the term "conjugate ratio" for $AC/BC$, and solves the problem: The conjugate ratio being given to divide the line [harmonically]. The procedure involves, as in the La Hire construction, the drawing of parallel lines and the laying off of equal lengths oppositely directed. Davies points out that admissible interchange of $C$ and $D$ as conjugates justifies the employment of that term; also the admissible interchange of $A,B$ and $C,D$ as pairs. He refers to Euclid vi, 16

$$AD \cdot BC = AC \cdot BD \tag{5}$$

or "the rectangle under the whole and middle part is equal to the rectangle under the extremes", and maintains that this (to him) inelegant property was not, as is sometimes claimed, the meaning of harmonic as employed by the ancient geometers. He asserts that harmonical is an "absurd" term to have been introduced into geometry. De Morgan, he tells us, suggested "consectors" for the "harmonic sectors" and recommends this nomenclature himself.

***Resumé.*** In terminating this historical account of the evolution of the cross-ratio construction, it is well to recall the purely projective method due to La Hire mentioned earlier in this paper. We shall

*Cajori, F., *loc. cit.*, p. 292.
†Coolidge, J. L., *loc. cit.*, p. 222.
‡Davies, T. S., *Ladies' and Gentleman's Diary* (1850), p. 92.

observe just how certain metric specializations of this construction, now almost uniformly used, furnish us with the first recorded attempts mentioned above. La Hire's construction follows: To construct on $l$ the harmonic conjugate $D$ of $C$ as to $A$ and $B$ we take a pair of lines (Fig. 6) on $A$, distinct and different from $l$. We meet them in P and $R$

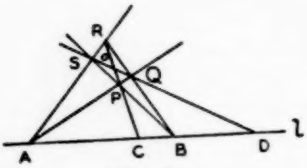



FIG 6

with a line on $C$. Join now $P$ and $R$ to $B$, meeting the lines on $A$ in $S$ and $Q$. Draw $SQ$, meeting $l$ in the required point $D$. For, $(ABCD) = (SQOD) = (BACD)$. But $(BACD) = 1/(ABCD)$. So $(ABCD)^2 = 1$. Now $(ABCD) \neq 1$, else there are duplicates among the points. Consequently, $(ABCD) = -1$, i. e., $D$ is the harmonic conjugate of $C$ as to $A$ and $B$. The uniqueness of $D$ completes the proof. We are thus led to a frequently used definition of four harmonic points in synthetic geometry; viz., "Four points $A, B, C, D$ on a line $l$ are said to be harmonic if there exists a quadrangle $P, Q, R, S$ of such a nature that a pair of opposite sides passes through $A$, another pair of opposite sides passes through $B$, and one each of the remaining pair of opposite sides passes through $C$ and $D$." This may be expressed by means of the equivalent statement: "Four points $A, B, C, D$ on a line $l$ are called harmonic if there exists a quadrangle $P, Q, R, S$ with $A$ and $B$ at two of its diagonal points and with those sides passing through the third diagonal point passing one through $C$ and one through $D$."

The Pappus construction presents a special case of the latter point of view. If we draw (shown in dotted lines Fig. 3) $AB'$ and $A'B$ meeting in $R$, then $RS$ will meet $l$ in the required point D. For, we have in $A', B', R, S$ a quadrangle with $A$ and $B$ as two of its diagonal points and sides $A'B'$ and $RS$ on the third diagonal point (at infinity in the direction $A'B'$) pass one on $C$ and the other on $D$. The quadrangle in this case is a trapezoid.

In similar manner the construction of La Hire falls in this category. Let us draw (shown in dotted lines Fig. 4) $AB'$ and $A'B$ meeting in $R$. Again, $RS$ meets $l$ in the required point $D$. The reasoning is exactly as given in the case of the Pappus construction. In this case the quadrangle has one pair of parallel sides, but lacks the symmetry of the former construction.

For the case $\lambda = -1$ the construction of Chasles (Fig. 5) duplicates that of La Hire. For the cases $\lambda \neq -1$, it falls in a different category. Indeed, if we draw $AB'$ and $A'B$ (shown in dotted lines Fig. 5) meeting in $R$, then $RS$ meets $l$ at $E$, the harmonic conjugate of $C$ as to $A$ and $B$; and not at $D$, for which $(ABCD) = \lambda$.

This more general case $\lambda \neq -1$ is, of course, just the case solved by Chasles noted earlier in this paper. We shall now outline what many regard as a more satisfying solution of the problem (Fig. 7). Given three points $A$, $B$ and $C$ on a line $l$ and a real number $\lambda$, to construct

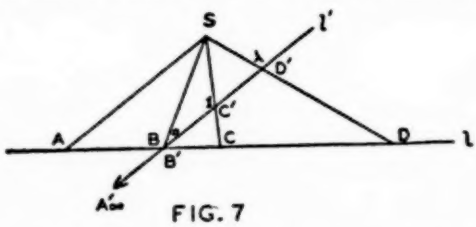



FIG. 7

a point $D$ such that $(ABCD) = \lambda$. Solution: Project $A$, $B$ and $C$ from $S$, not on $l$. On $B$ draw $l'$ parallel to $SA$, meeting $SC$ in $C'$. With $BC'$ as a unit lay off $BD' = \lambda \cdot BC' = \lambda$. Draw $SD'$ meeting $l$ in the required point $D$. For,

$$\begin{aligned}(ABCD) &= S(ABCD) = S(A'_\infty B'C'D') = (A'_\infty B'C'D') \\ &= (D'C'B'A'_\infty) = (D'C'B') = D'B'/C'B' = B'D'/B'C' = \lambda,\end{aligned}$$

where we have written $B'$ for $B$ as a point of $l'$.

We utilize here the concept of the cross-ratio as the basis of a projective coordinate system for points of a line. The numerical pattern is the fact that $(\infty 0 1 \lambda) = \lambda$. In effect, we replace a projective scale on $l$ by a metric one on $l'$, and make an abscissa $\lambda$ do the work of the cross-ratio. In Fig. 5 we have in effect such a metric scale on the line $A'B'$. The metric feature thus reduces us from the four line segments of the cross-ratio to the two line segments of the ratio. The choice of unit point reduces us finally to the single line segment $B'D'$, whose measure is our given real number $\lambda$.

## PART II—HARMONIC CONSTRUCTIONS

*Angle Bisectors.* One of the earliest instances of the harmonic separation is supplied by the bisectors of an angle of a triangle. Soon after beginning the study of plane geometry the student learns that (Fig. 8) $PA/PB = AC/BC = (ABC)$. In similar manner this ratio is equal to $(ABD)$. If we provide for the oppositely directed line seg-

ments by means of the minus sign, the ratio of these two ratios gives us $(ABCD) = -1$, perhaps the first instance of harmonic separation which one encounters in the study of mathematics.

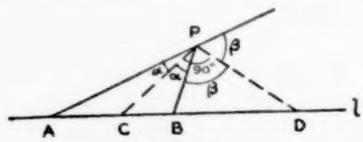



FIG. 8

It seems natural to ask: May we pass the other way from four harmonic points on a line to the bisectors of an angle of a triangle, i. e., may every harmonic set be regarded as having been obtained by the construction of bisectors as above outlined? Or, more explicitly: Given three points $A$, $B$ and $C$ on a line $l$, may we always utilize the bisector construction above as a method for locating $D$, the harmonic conjugate of $C$ as to $A$ and $B$? We shall see that the answer is in the affirmative; one may not only always use the bisectors, but also he may use them in many ways. One way is as follows: Take (Fig. 9)

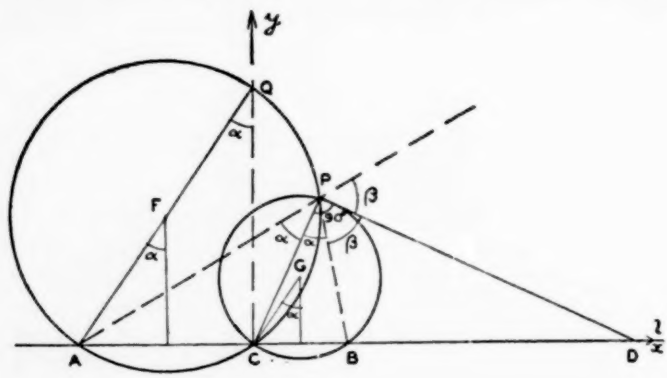



FIG. 9

the perpendicular bisector of $AC$ on $l$ and on it choose an arbitrary point $F$. With $F$ as a center and $FA$ as a radius draw a circle on $A$ and $C$. Now draw the perpendicular bisector of $CB$. On $C$ draw a parallel to $AF$, meeting this perpendicular bisector in $G$. With $G$ as a center and $GC$ as a radius draw the circle on $C$ and $B$. Let $P$ denote the second intersection of these two circles; and join $C$ to $P$. At $P$ draw a perpendicular to $CP$, meeting $l$ in $D$. Then $D$ is the desired harmonic conjugate of $C$ as to $A$ and $B$. We have in triangle $APB$ a solution of our problem; for, as we shall show, the interior and exterior

bisectors at $P$ play the same role as in Fig. 8. To show this we observe (Fig. 9) that $\measuredangle APC$ is equal to $\alpha$, the angle which $AF$ makes with the vertical, since the former is measured by one-half of twice the arc which measures the angle $\alpha$ at $F$. By construction, $CG$ makes the same angle with the vertical as that made by $AF$. In the manner just outlined $\measuredangle CPB = \alpha$. Consequently, $\measuredangle APB$ is bisected by $PC$. But $PD$ is constructed perpendicular to $PC$. Therefore, the exterior angle at $P$ is bisected by $PD$. Thus we have $(ABCD) = -1$, and since $D$ is unique we have solved the problem of utilizing the bisectors of an angle of a triangle to construct the harmonic conjugate.

*Two Circles.* Another construction for $D$, also involving two circles, but in this case with the first circle on $A$ and $B$, rather than on $A$ and $C$, follows. Draw any circle on $A$ and $B$. Let the perpendicular bisector of $AB$ at $M$ meet this circle in $N$, as shown in Fig. 10.

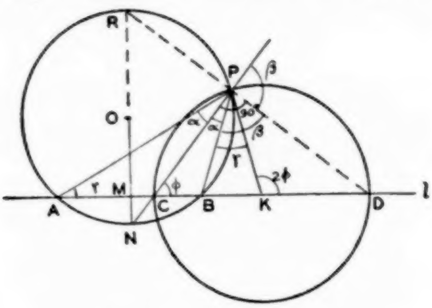



FIG. 10

Draw $NC$, meeting the circle $ANB$ in $P$. At $P$ draw a tangent to the circle $ANB$, meeting the line $l$ in $K$. The circle with $K$ as centre and $KP$ as radius passes through $C$ and meets $l$ again in the required point $D$. Proof: Join $P$ to $A$ and to $B$. Now $\measuredangle APB$ is bisected by $PN$, since $\measuredangle s$ $APN$ and $BPN$ are each measured by one-half of the equal arcs $AN$ and $BN$. If then, we can show that the circle with centre $K$ and radius $KP$ passes through $C$, we shall know that $\measuredangle CPD$ is a right angle, and that $PD$ bisects $\measuredangle APB$ externally, making $D$ the harmonic conjugate of $C$ as to $A$ and $B$. It remains to show that the circle with centre $K$ and radius $KP$ passes through $C$. Observe that $\measuredangle PAB = \measuredangle BPK = \gamma$, say, since each is measured by one-half the arc $BP$, and that $\measuredangle KCP = \measuredangle CPK$, since each is equal to $\alpha + \gamma$. This means that triangle $CPK$ is isosceles, and that $KC = KP$, i. e., the circle on $P$ with $K$ as centre passes through $C$.

*One Circle.* Although each of these constructions for passing from four harmonic points to bisectors employs two circles, it is clear

that in the case of the second solution the circle with centre $K$ is not needed. After locating $P$ as in that solution one has merely to draw at $P$ the perpendicular to $CP$. This perpendicular meets $l$ in the desired point $D$. This method has the advantage of employing one circle instead of two.

Another method employing only one circle consists in proceeding as above with the circle on $A$ and $B$, the perpendicular bisector meeting it in $N$, the determination of $P$, as there. Then instead of using the tangent $PK$, we have merely to join $P$ to $R$, the second intersection (Fig. 10) of the circle $ANB$ with the perpendicular bisector. This line $RP$ meets $l$ in the desired point $D$. For, $NP$ is the bisector of $\measuredangle APB$ and $\measuredangle RPN$ is a right angle, since $NR$ is a diameter. Of course, this solution is subject to the freedom of choosing the line $NC$ arbitrarily and then determining the circle as that on $A$, $N$ and $B$; instead of taking the circle on $A$ and $B$ arbitrarily and afterwards drawing $NC$. In both approaches, however, the perpendicular bisector is essential.

Still another way to accomplish* the harmonic construction with only one circle, in fact a semi-circle, is as follows: On $AB$ as diameter (Fig. 11) describe a semi-circle. Join any point $P$ of this semi-circle

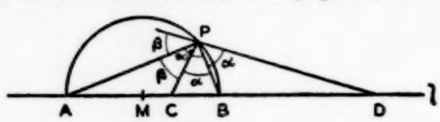



FIG. 11

to $C$ and to $B$. Draw through $P$ a line on the opposite side of $PB$ from $PC$, making an angle equal to the angle $BPC$. This line meets $l$ in the required point $D$. For, upon joining $P$ to $A$ and noting the right angle $BPA$, we are led to the bisectors of the angle $P$ of the triangle $DPC$. This solution shows $A$ to be the harmonic conjugate of $B$ as to $C$ and the $D$ just found. Hence $D$ is the harmonic conjugate of $C$ as to $A$ and $B$. Here we have used the conjugate features. The circle is the Circle of Apollonius.

*Inversion.* One may dispense with even this one circle, and replace it by an inversion. The familiar use of the circle in the method of inversion may be replaced by similar triangles. We take, as before, the perpendicular bisector of $AB$. On it we choose a point $N$. Now draw $NC$. On $NC$ produced locate $P$, the inverse point of $C$, using $NB$ as the constant of inversion (Fig. 12). To do this let us lay off on the sides of any convenient angle $O$ (Fig. 13) the lengths $OS = NC$,

*For this suggestion the writers are indebted to their colleague, Professor E. T. Browne.

$OT = OV = NB$. Draw $ST$. On $V$ draw the parallel to $ST$. This parallel meets $OT$ in $W$ such that $OW = NP$, the required length. For,

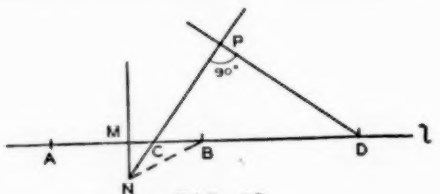



FIG. 12

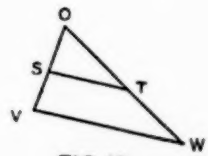



FIG 13

from similar triangles $NC/NB = NB/NP$; i. e., $NP$ is the third proportional of $NC$ and $NB$, which gives $NC \cdot NP = NB^2$, as is required. The length $OW$ laid off from $N$ along $NC$ produced gives us the desired point $P$. The perpendicular to $PN$ at $P$ now completes the determination of $D$ without using a circle at all. It remains to see that the point $P$ so located on $NC$ produced is the same point $P$ as that picked out by the circle $ANB$ of Fig. 10. To see this observe (Fig. 10) that $NC \cdot CP = AC \cdot CB = (AM+MC)(MB-MC) = (MB+MC)(MB-MC) = MB^2 - MC^2$. Consequently, $NC \cdot NP = NC(NC+CP) = NC^2 + NC \cdot CP = NC^2 + MB^2 - MC^2 = MN^2 + MB^2 = NB^2$. This identifies the point $P$ picked out by the circle $ANB$ with the point $P$ located by the inversion.

*Similar Triangles.* The method of similar triangles may be applied directly to the problem and furnishes us with a solution without resorting to an inversion. We proceed as follows: Take a line $m$ (other than $l$) on $A$. On $m$ take any length $AE > AC$. Draw $CE$. On $B$ draw a parallel to $CE$, meeting $m$ in $F$. On $A$ with radius $AE$ describe an arc. On $B$ with radius $EF$ describe an arc. These arcs meet in $P$ of Fig. 14, a point on the Circle of Apollonius. At $P$ a perpendicular

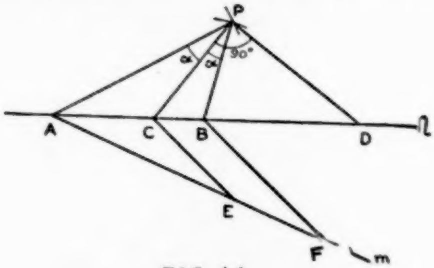



FIG. 14

to $CP$ meets $l$ in the desired point $D$. For, $AC/BC = AE/FE = PA/PB$. This shows that $PC$ bisects the angle $APB$. Consequently, the perpendicular to $PC$ at $P$ meets $l$ in $D$, the harmonic conjugate of $C$ as to $A$ and $B$.

The method of similar triangles may be applied to the solution of the problem without the implied use of the Circle of Apollonius, and without the use of the bisector property. Let $x$ denote the distance (Fig. 15) from $B$ to the required point $D$. We have

$$AD/x = (AB+x)/x = AB/x+1.$$

From which $$AB/x = AC/CB - 1 = (AC-CB)/CB.$$

Now lay off $AF$ on $l$ in Fig. 15 equal to $AC-CB$. We have $x = AB \cdot CB/AF$, and therefore $AF/AB = CB/x$; from which we see

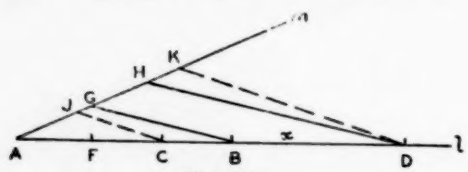


FIG. 15

that $x$ is the fourth proportional of $AF$, $AB$ and $CB$. On $A$ choose any convenient line $m$, other than $l$. On $m$ lay off $AG = AF$. Draw $GB$. On $m$ lay off $GH = CB$. On $H$ draw the parallel to $GB$. This line meets $l$ in the required point $D$. As a constructional check upon this determination we may on $m$ lay off $AJ = CB$, and draw $JC$. Then on $m$ lay off $AK = BD$. On $K$ draw the parallel to $JC$. This should pass through $D$, giving us the relation $AC/AJ = AD/AK$, or $AC/CB = AD/BD$, as required.

*A Special Problem.* Before concluding the synthetic treatment of the harmonic conjugate construction, we shall outline a solution of a remarkable theorem* closely related to it; namely: To divide a line-segment $AB$ harmonically in $C$ and $D$, being given the length $l$ of the segment $CD$. To accomplish this, draw any circle (Fig. 16) through

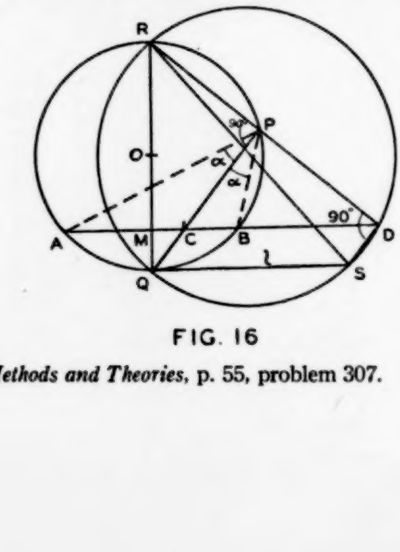


FIG. 16

*Petersen, J., *Methods and Theories*, p. 55, problem 307.

$A$ and $B$. Erect a perpendicular at $M$, the mid-point of $AB$, cutting the circle in $Q$ and $R$. Draw the tangent to the circle at $Q$, and lay off $QS=l$. On $RS$ as a diameter describe a circle, cutting $AB$ produced at a point which we shall denote by $D$, showing in the course of the proof that it is the point $D$ of the theorem. Join $D$ to $R$ by means of a straight line. Let $P$ denote the intersection, other than $R$, of this line with the circle $AQB$. Draw the line $QP$. We shall show that $QP$ meets $AB$ in the point harmonically conjugate to $D$ as to $A$ and $B$, and such that its distance from $D$ is the given length $l$; i. e., that we have located $C$. Proof: Consider triangle $APB$. Angles $APC$ and $CPB$ are equal, each being measured by one-half of the equal arcs $AQ$ and $QB$, respectively. Now $PD$ is perpendicular to $PC$ since $QR$ is a diameter of the circle $AQB$. Consequently, $PC$ and $PD$ are interior and exterior bisectors of the angle $P$ in the triangle $APB$. This makes $C$ the harmonic conjugate of $D$ as to $A$ and $B$. Again, since $RS$ is a diameter of the circle $QSD$, we see that $SD$ is perpendicular to $RD$, and therefore parallel to $QC$. Figure $QSDC$ is then a parallelogram and $CD=QS=l$.

## PART III—ANALYTIC METHODS

*Sector Containing an Angle.* We come now to some analytic solutions of the problem. Let us ask first: What is the equation of the circle (Fig. 9) on the two points $A$ and $C$ which contains the fixed acute angle $\alpha$? For this purpose take $C$ as origin, $AB$ as $x$-axis, and the perpendicular to $AB$ at $C$ as the $y$-axis. Let $Q$ be the point where the circle in question meets the $y$-axis. Since $\measuredangle ACQ$ is a right angle, $Q$ is the other end of the diameter passing through $A$. Our circle passes through the three points $A=(a,0)$, $C=(0,0)$ and $Q=(0,-a\cot\alpha)$, where $a=CA$, a negative quantity. The equation of such a circle is

$$x^2+y^2-ax+a\cot\alpha\cdot y=0. \tag{6}$$

In similar manner the equation of the circle on $C$ and $B$ which is to contain this angle $\alpha$ is

$$x^2+y^2-bx-b\cot\alpha\cdot y=0, \tag{7}$$

where $b=CB$, a positive quantity.

*The Circle of Apollonius.* The result of eliminating $\alpha$ from (6) and (7) by multiplying the former by $b$ and the latter by $a$ and adding is

$$x^2+y^2-\frac{2ab}{a+b}x=0, \tag{8}$$

a circle on

$$C, P \text{ and } D = \left( \frac{2ab}{a+b}, 0 \right),$$

This is the Circle of Apollonius. The equation (6) represents for each $\alpha$ a circle on $A$ and $C$ subtending an acute angle $\alpha$ along the major arc. As $\alpha$ varies we have in (6) the pencil of circles on $A$ and $C$. Similarly, as $\alpha$ varies we have in (7) a pencil of circles on $C$ and $B$. The circles of these two pencils are associated in a one to one way by means of equal values of $\alpha$. For each value of $\alpha$ there arises (Fig. 9) in this way a unique point $P$, the intersection other than $C$ of the circle on $A$ and $C$ with the circle on $C$ and $B$. As $\alpha$ varies taking on all real values $P$ describes the Circle of Apollonius (Fig. 10) given by (8). This circle is said to be "on" $CD$ and "about" $A$ and $B$. It meets $l$ in $C$ and again in the harmonic conjugate $D$ of $C$ as to $A$ and $B$.

*Circle on A and B.* The equation of any circle passing through the points $A$ and $B$ is readily seen to be

$$x^2+y^2-(a+b)x+\lambda y+ab=0, \tag{9}$$

where $\lambda$ is an arbitrary parameter. This circle for any value of $\lambda$ will meet the Circle of Apollonius at right angles, as may easily be seen by applying the well-known test for orthogonality, which is satisfied for every $\lambda$.

If the circle on $A$ and $C$ has been chosen and its partner circle on $C$ and $B$ taken with it to determine the point $P$ for a definite value of the parameter $\alpha$, then the slope $m$ of their radical axis $CP$ is determined. The circle (9) for a definite $\lambda$ will usually not pass through the point $P$ so located. A necessary and sufficient condition for this is (Fig. 10) that

$$N = \left( \frac{a+b}{2}, \frac{a+b}{2} m \right)$$

lie on circle (9). This gives us the condition

$$\lambda = \frac{(a-b)^2-(a+b)^2m^2}{2(a+b)m}. \tag{10}$$

Because of the orthogonality of circle $ANB$ with the Circle of Apollonius (8), the tangent to the circle $ANB$ at $P$ (Fig. 10) passes through the centre of the Circle of Apollonius. But the centre of the circle (8) lies on the line $l$. This identifies the point $K$ of Fig. 10 with the centre of the Circle of Apollonius, and $KP$ with the radius of that

circle. We have thus an analytic justification of the second of the constructions involving two circles.

*The One Circle Construction.* The fact that this second circle is unnecessary appears if we consider the point $P=(\bar{x}, \bar{y})$,

$$(11) \qquad \bar{x}=\frac{2ab}{(a+b)(1+m^2)}, \quad \bar{y}=\frac{2abm}{(a+b)(1+m^2)},$$

in which $NC$ (Fig. 10) meets the circle $ANB$. All that we need to do this is to observe that the line on $P$ perpendicular to $CP$; viz.,

$$(12) \qquad x-\bar{x}+m(y-\bar{y})=0,$$

meets $l$ in $D=\left(\frac{2ab}{a+b}, 0\right)$, for every $m$.

*The Radical Axis.* We have so far employed as parameter $m$, the slope of the radical axis $CP$. The equation of this radical axis in terms of $\alpha$, which characterizes it, is obtained upon subtracting equation (6) from equation (7), and is

$$(13) \qquad y=\frac{a-b}{a+b}\tan\alpha\cdot x,$$

giving us the relation

$$(14) \qquad m=\frac{a-b}{a+b}\tan\alpha.$$

By means of (14) we are able to change parameter from $m$ to $\alpha$.

*The Tangent at P.* Since the tangent to the circle $ABP$ at $P$ has been seen to pass through

$$K=\left(\frac{ab}{a+b}, 0\right),$$

its slope may be obtained from the coordinates of this point and those of $P$, as given by (11). This slope is $2m/(1-m^2)$, showing, as is obvious from the figure, that $\phi$, the angle of inclination of the tangent $KP$, is twice that of the radical axis $CP$. Indeed, the slope of this tangent $KP$ (Fig. 10) could have been more readily obtained by geometric considerations. We have $KC=KP$, so $\measuredangle KCP=\measuredangle CPK=\phi$, say. Therefore $\measuredangle DKP=2\phi$, and the trigonometric formula for the tangent of twice an angle gives at once $2m/(1-m^2)$ for the slope of

$KP$, since the slope of $CP$ is $m$. The equation of this tangent is consequently

$$(15)\qquad y=\frac{2m}{1-m^2}\left[x-\frac{ab}{a+b}\right].$$

*The Perpendicular Bisector.* The perpendicular $NM$ of $AB$ (Fig. 10) meets the circle $ANB$ again in

$$R=\left[\frac{a+b}{2},\ -\frac{(a-b)^2}{2(a+b)m}\right],$$

If we join $R$ to $P$, we obtain again the line (15), meeting the line $l$ at $D$. We thus justify analytically the second of the one circle constructions.

*The Parameter $\alpha$.* If we systematically introduce the parameter $\alpha$ in place of $m$ by means of relation (14), we obtain from (10)

$$(16)\qquad \lambda=(a-b)\cot 2\alpha.$$

For the coordinates of the point $P$ we have

$$(17)\qquad \bar{x}=\frac{2ab(a+b)}{(a+b)^2+(a-b)^2\tan^2\alpha},\qquad \bar{y}=\frac{2ab(a-b)\tan\alpha}{(a+b)^2+(a-b)^2\tan^2\alpha}$$

It is important to observe that equations (17), as well as the simpler equations (11), represent parametric equations, in $\alpha$ and $m$ respectively, for the Circle of Apollonius. In each case the parameter whether $\alpha$ or $m$ has a definite geometric interpretation.

The equation of the tangent $PK$ to the circle $ANB$ at $P$ is

$$(18)\qquad y=\tan 2\phi\left[x-\frac{ab}{a+b}\right].$$

where $$\tan 2\phi=\frac{2(a^2-b^2)\tan\alpha}{(a+b)^2-(a-b)^2\tan^2\alpha}.$$

The coordinates of $R$ are $\left[\dfrac{a+b}{2},\ -\dfrac{a-b}{2}\cot\alpha\right]$.

*Vector Solution.* We shall conclude this analytic discussion with a solution of our problem by means of complex numbers. Let us use the customary $z=x+iy$, where the $x-$ and the $y-$axes are chosen as formerly. Choose, for example, $C$ closer to $B$ than to $A$. At $A$ (Fig. 17) we have $z_1=a(a<0)$. At $B$ we have $z_2=b(b>0)$. On the axis of

imaginaries at $G$, a unit's distance downward, we have $z_3 = -i$. Construct now the circle $AGB$, meeting the axis of imaginaries again at $H$. We have there $z_4 = -abi(-ab>0)$. Join $A$ to $H$, and on the line so obtained lay off $HJ = -(a+b)$ and $JK = 1$. Draw $CJ$, and draw also the parallel to $CJ$ on $K$. This line meets $HC$ produced in $L$. From similar triangles it follows immediately that

$$CL = \frac{ab}{a+b}.$$

Now from $C$ on $l$ lay off $CI = CL$. With $I$ as a centre and $IC$ as a radius describe a circle. This circle meets $l$ again in $D$, the desired harmonic conjugate of $C$ as to $A$ and $B$. For, all that we require of $z$ at $D$ is that $(a\ b\ 0\ z) = -1$. From this we conclude that

$$z = \frac{2ab}{a+b}.$$

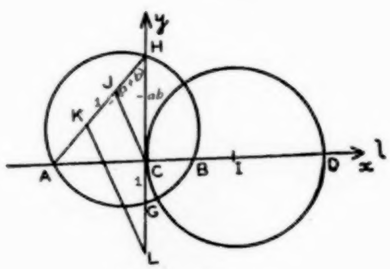


FIG. 17

But this is precisely what the construction above outlined secures for us. For the case where $C$ is closer to $A$ than to $B$, the above argument holds with the direction of the $x$-axis reversed. When $C$ bisects $AB$, the above construction gives $D$ as the point at infinity on the straight line through $A$ and $B$.

# The Study and Teaching of the History of Mathematics*

By U. G. MITCHELL
*State University of Kansas*

Recent scientific research has changed decidedly our perspective of civilization. We are beginning to recognize the forces in civilization which are constructive and those which are destructive. We are beginning to ask ourselves as never before how and why these forces have grown up and what they augur for the future of humanity. The history of civilization takes on new interest.

We have come to see that to study the history of a nation by delving into the records of the military campaigns in its wars and the diplomatic intrigues of its politics is comparable to studying the biography of an individual by examining the details of his hospital record during sieges of smallpox and scarlet fever. Smallpox and scarlet fever may condition or even destroy the physical life of a man, but they are not the constructive forces which build up within him the elements of preeminent character and power.

Fifty years ago the diseases of civilization loomed much larger in the study of human history than they do at present or are likely ever to do again. Some of us can still recall how valiantly we fought with Philip in his Macedonian phalanx and how adventurously we campaigned with Cæsar in Transalpine Gaul. Almost audible yet to our ears are the fife and drum in the American history textbooks of our youth; but the emphases of historical instruction are shifting rapidly from the pathological to the life-giving forces of civilization, and the history of science is coloring vividly the canvass.

*A paper presented as part of a symposium on the *Study and Teaching of the History of Science* at a joint session of Section L (Historical and Philological Sciences) of the American Association for the Advancement of Science, with the History of Science Society and the St. Louis Academy of Sciences, January 2, 1936, St. Louis, Mo.

The study of history may or may not aid in predicting future developments and in devising directly prophylactic measures for the improvement of civlization; but it certainly does furnish an understanding of the present that can be obtained in no other way. In all probability the time will soon be here when every educated person will be expected to understand the part played by science in general in the development of civilization and every specialist will be expected to have studied also the evolution of knowledge in his particular field.

There are some reasons why mathematics might naturally be expected to be the first scientific field to have its evolution studied and taught. It is the oldest, the simplest and the most exact of the sciences. Its accomplishments are precise and definite, and many of the milestones in the course of its progress are as clearly recognizable as the markers on one of our transcontinental highways. Accordingly, it should not be surprising to find that many colleges and universities are offering courses in the history of mathematics without giving to students majoring in astronomy, physics, chemistry, geology or medicine similar opportunity to study the great contributions of their particular fields to the development of civilization.

At least one writer on the history and philosophy of science, Mr. J. W. N. Sullivan, has gone considerably beyond Immanuel Kant's familiar dictum that each natural science is real science only in so far as it is mathematical. In his little book *The Tyranny of Science* Mr. Sullivan says:

> The fact is that science was undertaken as an intellectual adventure; it was an attempt to find out how far nature could be described in mathematical terms.

If we agree with Mr. Sullivan it becomes easy to believe that the study of the history of mathematics has much to contribute to the scientific perspective of anyone who studies it. Even if one does not agree with either Kant or Sullivan, he is probably willing to admit that mathematics penetrates so far into all scientific work in general that the study of the history of mathematics will throw many revealing sidelights upon the history of these other fields. Indeed, any serious student of the history of mathematics is well aware that he is studying only one phase, although a central one, of the history of science *in general* which makes up so large a part of all the history worth studying at all.

The study of history makes great demands upon its serious disciples. Not only must the human qualities of industry, patience, fairmindedness and critical judgment be cultivated, but also languages must be studied, old forms, words, symbols and abbreviations must be

learned, and some mastery of the criteria for weighing documentary evidence, both external and internal, and determining the relative values of source and secondary material, must be gained.

In addition to these general requirements, the study of the history of mathematics offers difficulties of its own. The mathematician deals with eternal verities. The right-angled triangle property, generally identified with the name of Pythagoras, may have been known to the Babylonians a thousand years before Pythagoras was born; but it states a relation as true before the birth of the first Babylonian as after the death of the greatest of the Greeks—a relation, in fact, independent of time and place.

Just as a master in any natural science does not need a knowledge of certain relations in order to verify their validity by experimentation, so the mathematician needs no knowledge of the origins of geometry in order to demonstrate its theorems. The masters of natural science assume the constancy of nature and the mathematician assumes the universality of logical inference. It has, therefore, seemed natural for all scientists, including mathematicians, of course, to feel that there was no scientific necessity for studying the histories of their special fields; but there is a human necessity—they need to understand the evolution of science itself in order to know its implications as they affect the destinies of humanity.

In one sense history is the antithesis of mathematics. History as a science and in so far as it is a science, presents the evidence and rests its case. It establishes no necessary conclusions that new evidence may not change tomorrow. Mathematics accepts *only* necessary conclusions. It tolerates no others. There may be various methods of rigorously showing the necessity of these conclusions, but the method is incidental to the conclusion. This antithesis is, however, more one of appearance than of reality, for the historian cannot escape interpretation entirely and the mathematician can neither escape his assumptions nor the changing conceptions of rigorous demonstration. Any worth while study of mathematics tends to form ideals of precision and inference adaptable to the verification of historical facts; and the history which does nothing but catalogue facts is about as interesting and serves much the same purposes as a dictionary.

These considerations bring us face to face with the paradox of historical study: the free use of imagination in historical investigation is both indispensable and inadmissible. Does any one think it is easy to rid his mind of its customary assumptions of life and ways of thinking and transplant himself into another age and another country so fully as to understand that other time and place and people?

The most difficult thing to do in attempting to comprehend any phase of human history relating to another generation is to acquire an adequate conception of the surroundings and circumstances of the past age—to reconstruct, as it were, the mental as well as the physical furniture (in imagination, of course) of the period under consideration.

The idea deserves emphasis.

Perhaps an incident I have heard may serve that purpose.

A man who had come to America as a poor immigrant had been so busy making a living that he had only very limited opportunities for education. Determined that his son should not be so handicapped, he saw that his son attended high school. One evening in the home the following conversation is reported to have taken place:

"Son, what you studying?"

"General History."

"General History—what is that?"

"Oh, it is all about what has happened in the world. Just now I was reading about Julius Cæsar."

"Julius Cæsar—who was he?"

"He was a famous Roman general. He once conquered a country so quickly that he merely sent back a message of three words: *veni, vidi, vici,*—came, saw, conquered."

"Well, Julius Cæsar was a fool."

"Oh, no, father. He was no fool. He was one of the greatest men who ever lived."

"I say he was a fool. Any man who will send three words when he can send ten for the same money is a fool."

We smile at the incongruity of placing the magnetic telegraph nineteen centuries before its time, but men have ventured to write histories who have made mistakes almost as serious.

In order to know the past as it was, we must see the men and women of the past as they were at that time. We must be able, in some degree at least, to think as they thought, feel as they felt, understand as they understood, be under such limitations as they were under. The imagination must have free range. It must be active enough to reconstruct for us in great detail scenes of the past. But for many such details there is no authentic record or description and modern scientific history demands that only such facts as have been carefully verified and meticulously documented can be put into the picture. It is as if the story-teller were throwing upon the silver screen of our mental vision pictures of the past which fascinate and thrill us when the scientific historian steps up beside us and insists upon our noticing that the technique of the picture is faulty, that the costumes are not quite accurate in some details and the stage properties slightly out of place, until our attention is distracted from players and action and we lose both interest and pleasure.

Possibly the solution of the paradox lies in making use of the imagination as freely as we please, but carefully making note of the time and place of passing from the realm of fact to the realm of fiction. There is joy in romance, but romance should not be allowed to masquerade as history. On the other hand, history must not be allowed to lose its human and dramatic elements. Mathematics has had its thrilling conquests and the names of Euclid and Pythagoras are familiar to millions today who never heard the names of conquering despots more dominating in that ancient time than Mussolini is today.

For purposes of the moment may we distinguish between the *story* of mathematics and the *history* of mathematics.

The *story* of mathematics in its entirety would include everything—events, thoughts, experiences and inventions—which have entered into its development. It would tell of difficulties encountered and ingenious ways of mastering them; of patient toil and struggle to capture strongholds of ignorance; of astonishing conquests in measuring the speed of light and weighing distant planets by the power of human reason. It would be dramatic and interesting but it can never be fully told. Most of it is gone beyond recovery.

The part of that story which has persisted in tangible physical objects or written records we call *history*. Its first considerations are accuracy and reliability. It demands definite authenticity. It may contain human and dramatic elements, but they must remain subordinate. And yet it would seem as if history should be the immortal soul of the story, which has lived and died and risen again to something of eternal life.

The first historians *were* story-tellers. Their stories were not always reliable but they inspired to noble and heroic deeds. The modern historian can be a story-teller and a reliable and accurate one. David Eugene Smith showed that the two could be one when he wrote his little book *Number Stories of Long Ago*—a book which has probably been read by more Americans than any other book on the history of mathematics ever published. Well aware of the double role he was playing and the difficulties in reconciling the two characters, he wrote two prefaces to the book—one for the grown-ups, in which he acknowledges the responsibility for historical accuracy, and one for the children, "Just between us and worth reading," in which he points out that the stories possess unusual interest because they are not merely make-believe stories but stories of what really may have happened when the world was young; and then, well-knowing that he had been as careful as he could be to make every detail as historically accurate as possible, he cannily adds: "Is this history? Never mind. What is

history but a story and is not every story a history of something? Why bother our heads over history? For us the story is the important thing."

The teaching of courses in the history of mathematics in colleges and universities is a comparatively recent development. It probably began in many places as it did at the University of Kansas 29 years ago as part of the preparation of prospective high school teachers of mathematics.

The teaching of any subject may be analyzed broadly, as a problem of human activity under three heads:

I. The aims or purposes to be accomplished.
II. The choice, organization and use of material in accomplishing these purposes through the medium of the teaching process.
III. The testing of students to determine whether or not, and to what degree, the teaching process has accomplished the ends sought.

So far as the problem of teaching the history of mathematics is concerned, some study and writing has been devoted to the first of these three considerations, but very little, if anything, to the other two. It would be interesting to know what would form a consensus of opinion of the professors teaching the history of mathematics as to the purposes they seek to accomplish by such teaching. The following are some of the aims likely to receive mention.

1. *To enrich the student's knowledge of the mathematics whose history is studied.* In tracing the development of mathematical ideas, a student is likely to gain a more complete understanding of those ideas and to comprehend better their relations to other ideas developing at the same time. They may even discover that two ideas which they had thought unrelated are only different aspects of a third more general idea.

2 *To aid in teaching.* The theory of recapitulation holds to some extent mentally as well as physically. The child repeats the development of the race and the evolution of mathematics offers valuable suggestions both as to choice of material and use of that material. About thirty years ago various teachers and textbook writers began to feel that students could be given something of the historical background of the mathematics they studied. Accordingly, historical notes were inserted in textbooks, as, indeed, had been done by some textbook writers in other countries centuries ago. Curiously enough these insertions did not arouse as much enthusiasm on the part of students as was expected. The addition of biographical sketches and portraits of eminent mathematicians secured in a way, greater attention. It was fascinating to see what marvelous transformations could be wrought in those portraits by a few artistic strokes of pen or pencil. The

merest novice could easily change the delicate Descartes into a perfect pirate; a long flowing beard to match the equally long and curly locks of Leibniz accredited him to Russia instead of to Germany; and the gentle and refined countenance of the philosophic Newton altered astonishingly with the addition of an upturned German moustache. Teachers are now aware that the most effective use of the history of mathematics is by the teacher who knows the history well and knows when and how to use it interestingly.

3. *To exhibit mathematics as a living, growing and changing science.* Most people who have never given much thought to the subject take for granted that mathematics has always been just as it is now. To see the various forms in which the same ideas have been expressed by different peoples and to see how long and painful has been the struggle to master certain ideas (such as limit, for example) is a revelation to most students.

4. *To enable students to comprehend better the part that mathematics has played in the development of sciences in general and in the advancement of civilization.* A famous Doctor of Medicine is often quoted as having said that medicine would become a science when doctors learned to count; but it is probably as near the truth to say that science itself began when men learned to count. Certain it is that much of our modern civilization could not be developed, at least in the way in which it was developed, until certain developments in mathematics had taken place. There is as much truth as wit in Professor Crathorne's remark in closing a discussion touching upon this point: "And men go up in flying machines made of wood, wire, complex fractions, gasoline and exponents."*

5. *To exhibit movements in the past development of mathematics which may suggest lines of development in the future.* A considerable part of the development of mathematics has grown out of the needs of other fields. Physics, chemistry, engineering, natural sciences have made great use of metric mathematics. The social sciences have not made comparable progress. They have tried to use metric mathematics when it is likely that their need is for non-metric mathematics—the mathematics of transformations, invariants, covariants, topology and field theory. Certain psychologists at the present time are attempting to make use of field theory in the development of their field.† The

*University of Illinois State High School Conference, November, 1915. See *School Science and Mathematics*, vol. 16, page 431.

†See, for example, J. F. Brown's *Psychology and the Social Order: An Introduction to the Dynamic Study of Social Fields* (McGraw-Hill, N. Y., 1936). Especially Appendix A (pp. 469-486) on *The Mathematical and Methodological Background of Psychological Field Theory.*

needs of their work may lead to new developments in mathematics or to such new uses of mathematics already developed as may have great influence upon future progress in all social science fields.

Various objectives other than the five I have mentioned are in the minds of those teaching the history of mathematics. I wonder how many of them have thought of the following reason given by Robert Potts nearly 100 years ago as his objective in beginning his *Elements of Geometry,* with a forty-page historical introduction: "respecting the history of science it has been remarked that it serves, at least, to commemorate the benefactors of mankind; an object which can scarcely be considered as unworthy or unimportant."

Concerning the organization of courses and the testing of results, I do not venture to speak. In spite of the fact that many colleges and universities are now offering courses in the subject, there has been no concerted effort to standardize the work or devise objective tests. Perhaps none is needed. It might be possible to improve the work more by informal discussion and helpful exchange of ideas than by appointing committees and sending questionnaires. I do not know, but I am confident that progress is being made, workers in the field are increasing, books and memoirs are multiplying and source material is becoming more accessible. It seems reasonable to expect that new and now unsuspected ways will be found for using the history of mathematics to connect mathematics with life and to make the study of elementary mathematics a natural growth in ideas about number and form.

---

All our scientific hypotheses must be shot through in all directions by the peculiarities of the mental constitution of those who build up this science. Thus the latter is deeply colored not merely by individual, but by racial characteristics.

Great achievement is intimately bound up with spontaneity of interest and effort. Science truly is international and profits much by the interaction of one people with another, even to the extent of obtaining fundamental values in this way. But the primary need is that each nation shall set about its part of the common problem in a way of its own, allowing its national individuality full freedom for self-expression according to its peculiar bias. Otherwise its attainments are a mere appendage to those of another people and contribute but little to the general good which would not have been as well developed if that nation had never existed. And so it has come about justly that history has recorded permanently but little concerning any civilization which did not develop from within. Likewise no national contributions to science can be of great and abiding interest unless they spring up essentially from the life and thought of the nation itself.—R. D. Carmichael, *Individuality in Research, Scientific Monthly,* December, 1919, pp. 523, 525.

# *The Teacher's Department*

*Edited by*
JOSEPH SEIDLIN and JAMES McGIFFERT

## Note on the Method of "Moment Areas"

By H. R. GRUMMANN
*Washington University*

The great utility of Professor Greene's Method of Moment Areas* is conceded by all who are concerned, in the subject of mechanics of materials, with the deflection of a loaded beam.

The method consists in the application of two theorems, the derivations of which in certain textbooks leaves something to be desired. At any rate, the writer has found that in applying the second theorem, students often calculate the moment of the area between the bending moment graph and the $x$-axis, about a vertical through the wrong end of the bending moment diagram. This may be attributable, in part, to a somewhat vague derivation of the theorem in the textbook. For the sake of completeness, derivations of both theorems are made, atlthough published derivations of the first theorem are generally satisfactory.

In the figure, $s$ represents arc length measured along the elastic curve. The axes are chosen so that $s=0$ when $x=0$, and $s$ increases as $x$ increases. If tangents are drawn to the elastic curve where $s=0$ and where $s=l$, the angle of contingence is $(\sigma_l-\sigma_0)$ and the average curvature of this portion of the beam of length $l$ is $(\sigma_l-\sigma_0)/l$. Between two points on the elastic curve $\Delta s$ units apart, if the corresponding angle of contingence is $\Delta\sigma$, the average curvature is $\Delta\sigma/\Delta s$ and the curvature at a point is the limit of this, viz., $d\sigma/ds$. Also,

$$\frac{d\sigma}{ds}=\frac{d^2y}{dx^2}\cos^3\sigma$$

where $\sigma$ is the inclination of the tangent line drawn to the elastic curve at the point in question.

*Parcel and Maney, *Statically Indeterminate Stresses*, pp. 40-4, 2nd ed., Wiley, 1936.

If $M$ is the bending moment at any point on the beam, we start with the fundamental relation

$$M = EI\frac{d\sigma}{ds} .$$

Separating variables and integrating between the proper limits we have

$$\int_0^l M ds = EI \int_{\sigma_0}^{\sigma_l} d\sigma$$
$$= EI(\sigma_l - \sigma_0).$$

This equation is exact and may, for example, be used to calculate the angle of contingence between two points on a strongly bent beam like a clockspring.

If, however, from $s=0$, to $s=l$, the value of $\sigma$ at all points on the curve is nearly equal to zero, $s$ may be identified with $x$, and the integral on the left hand side of the equation becomes

$$\int_0^L M dx = A$$

where for $l$ we have written $L$, the distance between the two points on a nearly "horizontal" beam for which the corresponding angle of contingence is being considered or calculated. This integral is the area between the bending moment diagram and the $x$-axis bounded by two "verticals" (perpendiculars to the $x$-axis) at $x=0$ and $x=L$. This is the content of Theorem I, which states that the angle of contingence between any two points of a beam is the area of the bending moment diagram between the corresponding two verticals, provided that the inclinations of all tangents to the elastic curve between the two points in question are approximately zero.

If a piece of string of length $l$ lying along the elastic curve from the point where $s=0$ to the point where $s=l$ is unwound from the point where $s=l$, a portion of length $p$ of an involute of the elastic curve is traced out by the moving end of the string. This length of arc $p$ of this involute may be called a deflection of the beam and for beams of small curvature may be identified with the length of a perpendicular (not shown in the figure) from $Q$ to the tangent to the elastic curve at $P$.

Using primes to denote the involute one readily sees from the figure that for any point $(x', y')$ on the involute the corresponding point

on the elastic curve is $(x,y)$. $\xi$ is the radius of curvature of the involute at the point $(x',y')$ and represents the unrolled portion of the string at the instant the point $(x',y')$ is described. The tangent to the involute at $(x',y')$ (with inclination $\sigma'$) is perpendicular to the tangent to the elastic curve (with inclination $\sigma$) at the point $(x,y)$. Now

$$\xi = \frac{ds'}{d\sigma'},$$

the definition of a radius of curvature.

But $d\sigma' = d\sigma$, on account of the mutual perpendicularity of tangents at corresponding points on the elastic curve and on the involute. Hence, separating the variables and integrating between the proper limits

$$\int_{\sigma_0}^{\sigma_l} \xi d\sigma = \int_0^p ds', \quad \text{or}$$

$$\int_0^l \xi \frac{d\sigma}{ds}\, ds = p.$$

But

$$\frac{d\sigma}{ds} = \frac{M}{EI},$$

Hence (if $EI$ is constant)

$$\int_0^l \xi M ds = EIp.$$

Again, this is an exact formula for a strongly bent beam. But with the usual assumption that $\sigma$ is approximately zero, $s$ may be identified with $x$, $\xi$ with $(L-x)$, and we have

$$EIp = \int_0^L \xi M dx$$

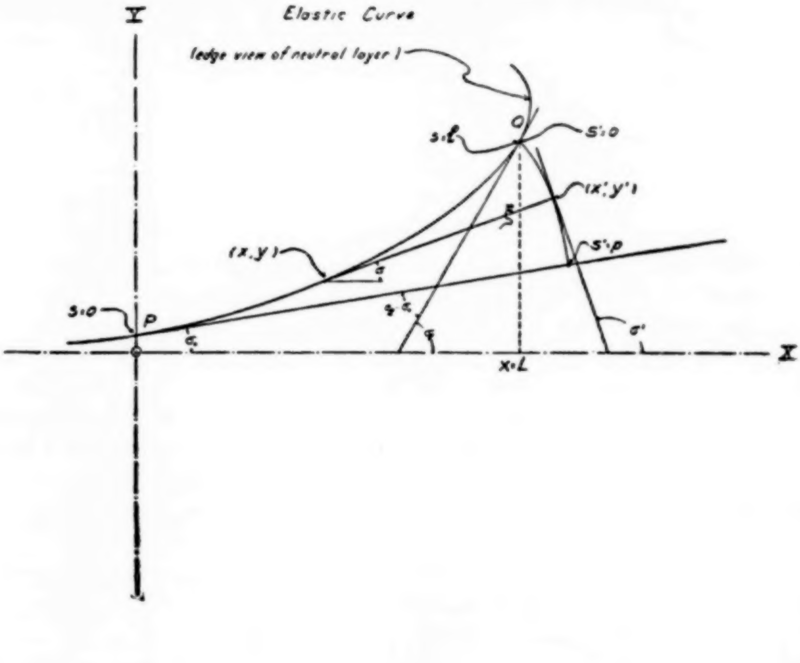
$$= \int_0^L \xi M d\xi,$$

the moment of the area of the bending moment diagram about that one of its two bounding verticals corresponding to the place on the

beam where the involute of length $p$ is traced out. This moment of area of the bending moment diagram about one of its bounding verticals is equal to its area times $\bar{\xi}$ and hence

$$EIp = A\bar{\xi}\,.$$

This is Theorem II of the Area-Moment Method. The coordinate $\bar{\xi}$ of the centroid of the area $A$, is often determinable by inspection, which makes the application of Theorem II comparatively simple.



# The First College Course in Algebra

By T. M. GORSKI and WILLIAM H. POWERS
*Alliance Junior College, Cambridge Springs, Pa.*

One of the fundamental objectives in all courses is "to train the student to meet new situations." This presupposes adequate drill work to assure mastery of principles and some practice in the classroom in meeting such "new situations." It is the belief of the writers that the first college course in algebra, as at present constituted in most institutions, fails to meet these objectives fully. Not only do teachers of advanced mathematics courses complain that students come to them without mastery of material they have "covered" but our experience in teaching such first year courses over a number of years convinces us that the physical limitation of time will not permit coverage of all first course material even in a four hour per week course of one semester. An analysis of the topics covered by nine well-known texts examined reveals that there are 20 major topics divided into 122 subdivisions, i.e., less than one hour average for each topic in a 72 hour course. This allows no time for "enrichment" material, for applications to the sciences, business, and statistics, or for digressions on the history of mathematics, recreational mathematics, and biographical studies of great mathematical geniuses and inventions—all of which seem desirable if the course is to have vitality and is to inculcate social attitudes and interests.

We believe, therefore, that the first college course in algebra needs revision downward in the number of topics taught and upward in the time spent on the vitalizing, drill, and "new situation" requirements. To this end we sent a form containing all the topics listed in the nine texts on *First College Algebra* to 56 institutions representing major universities, leading liberal arts colleges, and a few junior colleges, asking the departments of mathematics to classify items listed as essential, desirable, and undesirable, for the course. Replies were received from 32 institutions and are summarized by topics in Graph I— which shows percentages of total replies favoring 50 per cent or more of each article. It is interesting to note that there is nearly unanimous agreement on only points C to K inclusive, good agreement on all others except N, O, P, and T. This suggests that teachers look upon all topics as important with the exception of the four above mentioned (N, O, P, and T). Analysis by subtopics shows that 63

per cent of the reporting institutions consider the following subtopics non-essential. (Table I):

Item 15. Highest Common Factor in Fractions.
" 23. Determinants of 2nd and 3rd Order as Applied to a System of Simultaneous Equations (linear).
" 59. Harmonic Progression.
" 68. Logarithms to Base *e*.
" 69. Change of Base of Logarithms.
" 70. Graphs of Logarithmic and Exponential Function.
" 71. Exponential Equations.
" 72. Problem Applications of Exponential Functions.
" 73. Partial Fractions.
" 74. Limit of a Sequence.
" 75. Convergence of an Infinite Series.
" 76. Infinite Geometric Series.
" 77. Binomial Series.
" 78. Tests for Convergence and Divergence Positive Terms.
" 79. Ratio test.
" 80. Series of Positive and Negative Terms.
" 81. Power Series.
" 82. Compound Interest.
" 83. Ordinary Annuities (Present Value and Compound Amount).
" 84. Amortization and Sinking Funds.
" 85. Bonds.
" 86. Interest (Effective and Nominal Rates).
" 92. Binomial Theorem—Proof by Mathematical Induction.
" 93. Binomial Coefficients.
" 95. Graphical Solution of Inequalities in One Variable.
" 96. Definitions and Algebraic Operations with Complex Numbers.
" 97. Geometric Representation of Complex Numbers.
" 98. Trigonometric Form of Complex Numbers.
" 99. DeMoivre's Theorem.
" 107. Limits of Real Roots.
" 109. Newton's Method for Irrational Roots.
" 110. Horner's Method for Irrational Roots.
" 111. Illustrations of Solutions of General Cubic and Quartic Equations.
" 112. Inversions.
" 113. Definition of a Determinant of Order 3.
" 114. Definition of a Determinant of Order $N$.
" 115. Sigma Notation.
" 116. Expansion by Minors.
" 117. Sum of Determinants.
" 118. Evaluation of Determinants.
" 119. Systems of $M$ Linear Equations in $N$ Unknowns When $D \neq 0$ and When $D = 0$.
" 120. Homogeneous Linear Equations.
" 121. Systems of $M$ Linear Equations in $N$ Unknowns Where $M \neq N$.
" 122. Sylvester's Dialytic Method of Elimination.

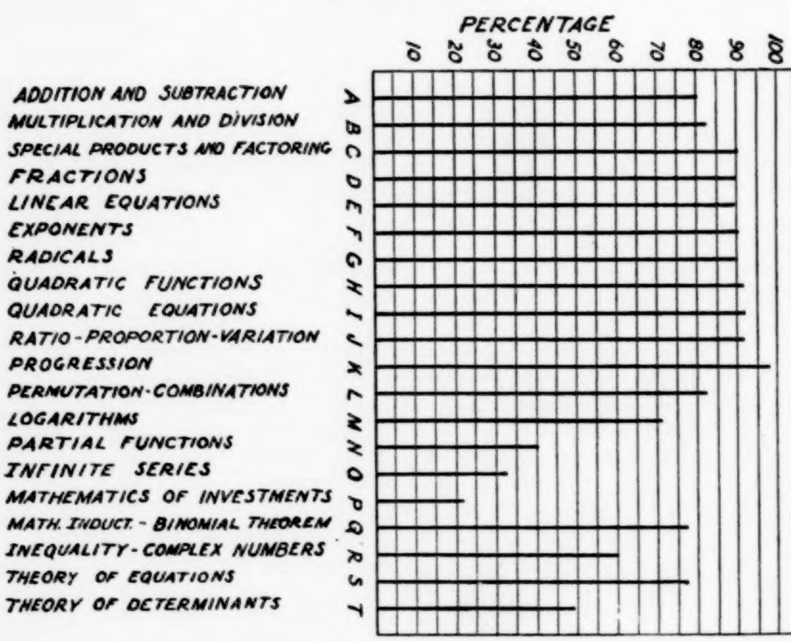
The number of subtopics considered essential by 63 per cent of reporting institutions was 85 out of 122. This paper suggests revision which may be made to permit sufficient coverage of essentials together with adequate drill, new situation problems, and vitalizing material.

*Conclusions*:

(1) The first course in College Algebra may be revised to include eighty-five subtopics without sacrificing material deemed essential by 63 per cent of representative, reporting institutions.

(2) Revision will permit time for additional topics at the discretion of the instructor and will permit the introduction of more original problems, historical material, and some elementary treatment of the philosophy of numbers.

GRAPH I



# *Mathematical World News*

*Edited by*
L. J. ADAMS

Dr. Saunders MacLane has been appointed Assistant Professor of Mathematics at Harvard University beginning September 1, 1938

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland on Saturday, May 7, 1938. The following five papers were read:

1. *The relation of the range of a sample to the standard deviation of the population.* Dr. L. S. Dederick, Aberdeen Proving Ground.

2. *Some naval tactics in vector analysis.* Professor C. H. Rawlins, Jr., Postgraduate School, U. S. Naval Academy.

3. *Infinite continued radicals and iteration of polynomials.* Mr. Aaron Herschfeld, Social Security Board, Washington, D. C.

4. *Mathematical applications common to practical meterology.* Lieut. H. B. Hutchinson, Postgraduate School, U. S. Naval Academy.

5. *Partitio numerorum.* Professor Hans Rademacher, University of Pennsylvania.

At Indiana University Professor S. C. Davisson has retired after 48 years of service in the mathematics department; Dr. Emil Artin, who came to the University of Notre Dame from the University of Hamburg in the fall of 1937, has been appointed Professor of Mathematics; Dr. Agnes E. Wells, who has been Dean of Women in addition to teaching mathematics, has given up the work of dean to give full time to teaching as Professor of Mathematics; Professor K. P. Williams has been appointed Chairman of the department.

The William Lowell Putnam Prize Scholarship for 1938 has been awarded to Mr. I. Kaplansky of the University of Toronto. This scholarship is awarded annually by the division of mathematics at Harvard University for study at that university to one of the first five contestants in the William Lowell Putnam Mathematical Competition. Mr. Kaplansky plans to use this award during the academic year 1939-40.

Professor W. M. Whyburn, chairman of the mathematics department of the University of California at Los Angeles, announces

the following additions to his department: Dr. T. Y. Thomas (professor), and Dr. A. E. Taylor (instructor).

The University of Illinois recently published the second volume of the *Collected Works of Professor G. A. Miller*, which is mainly devoted to his publication from 1900 to 1908 but contains two articles written especially for this volume. The former of these relates to the history of group theory during the period covered by this volume, while the latter is devoted to primary facts in the history of mathematics and deals mainly with questions of elementary mathematics and with errors in its history.

The American Mathematical Society will meet in Cleveland, Ohio on November 25-26, 1938, and at Los Angeles, California on November 26, 1938. At the Cleveland meeting Professor C. C. MacDuffee will read a paper: *Modules in Algebraic Fields*, and Professor V. G. Grove will read one on *A tensor analysis for a* $V_k$ *in a projective space* $S_m$.

During the past summer Professor James McGiffert, Rensselær Polytechnic Institute, addressed the Los Angeles Breakfast Club on the subject of the 200 inch telescope being constructed on Palomar Mountain, in Southern California, and what this telescope would reveal in a trip to the moon. In Baton Rouge Professor McGiffert addressed the faculty and graduate students of the Louisiana State University on *Streamlining Through Interstellar Space*. In Chicago, he was the principal speaker at a banquet of the Rensselær alumni.

# *Problem Department*

*Edited by*
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

## SOLUTIONS

No. 164. Proposed by *V. Thébault*, Le Mans, France.

A line $L$ cuts the sides $BC$, $CA$, $AB$ of a triangle $T$ in the points $\alpha,\beta,\gamma$. Show that

(1) The parallels to $BC$, $CA$, $AB$ through the centers of the circles circumscribed upon $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ determine a triangle $T'$ symmetrically equal to $T$.

(2) The center of symmetry of the triangles $T$ and $T'$ is situated on the Newton line of the quadrilateral $ABC, L$.

Solution and generalization by the *Proposer*.

(1) Let $O,O_a,O_b,O_c$ be the circumcenters, $G,G_a,G_b,G_c$ the barycenters, and $H,H_a,H_b,H_c$ the orthocenters of the triangles $T=ABC$, $T_a=A\beta\gamma$, $T_b=B\gamma\alpha$, $T_c=C\alpha\beta$, respectively; $\delta$ the Newton line of the complete quadrilateral $Q=(T,L)$ (or the line of midpoints $M,N,P$ of the diagonals $A\alpha,B\beta,C\gamma$).*

(a) Through the vertices of the given triangle draw parallels to any arbitrary direction which cut the line $L$ in the points $A'$, $B'$, $C'$. The parallels to the sides $BC$, $CA$, $AB$ through the points $A'$, $B'$, $C'$ determine, by their intersections two by two, a triangle $T_1=A_1B_1C_1$ symmetrically equal to triangle $ABC$.†

To demonstrate this theorem, we need only observe that if the line $L$ undergoes an arbitrary translation, the triangle $T_1$ is subjected to the same translation. This permits us, in effect, to take the line $L$ through the vertex $A$ of triangle $T$. In this manner, the vertex $A_1$

*See Math. Gazette, 1934, p. 200; 1922-3, p. 59, 312.—Ed.
†J. Neuberg, Wiskundig Tydsschrift, X, p. 80.

of triangle $T_1$ lies on the side $BC$ of triangle $T$ and the property results from the fact that the relative heights of the vertices $A$, $A_1$ of the two triangles are equal with contrary sign.

(b) There results the relations:

$$A'\beta/A'\gamma = A'B/A'C, \quad B'\gamma/B'\alpha = B'C/B'A, \quad C'\alpha/C'\beta = C'A/B'C$$

so that if the direction of the parallels $AA'$, $BB'$, $CC'$, varies, the points $\alpha, \beta, \gamma$ are the homothetic centers of similar points cut out upon the line $L$ by the points $B'$ and $C'$, $C'$ and $A'$, $A'$ and $B'$. The points $A_1, B_1, C_1$ describe then similar points upon the lines $\delta_a$, $\delta_b$, $\delta_c$ passing through the points $\alpha, \beta, \gamma$, respectively. The vectors $\overline{B_1C_1}$, $\overline{C_1A_1}$, $\overline{A_1B_1}$ being equipollent to the vectors $\overline{CB}$, $\overline{AC}$, $\overline{BA}$, the lines $\delta_a, \delta_b, \delta_c$ are parallel.

Moreover, the same lines are parallel to those joining the points $M$, $N$, $P$ to the center of symmetry $S$ of the triangles $T$, $T_1$; the point $S$ is then on the line $\delta = (M,N,P)$ and it itself is parallel to the lines $\delta_a, \delta_b, \delta_c$.

(c) The orthocenters $H$, $H_a$, $H_b$, $H_c$ of the triangles $T$, $T_a$, $T_b$, $T_c$ are on the orthocentric line of the quadrilateral $Q = (T, L)$ and the segments $AH_a$, $BH_b$, $CH_c$, perpendicular to $L$, are parallel.

Then, from (a) and (b), the parallels to the sides $BC$, $CA$, $AB$ of the triangle $T$ through the points $H_a$, $H_b$, $H_c$ determine a triangle $T' = A_hB_hC_h$ symmetrically equal to $T$.

(d) The points $M$, $N$, $P$ being the barycenters of the pairs of opposite vertices $A$ and $\alpha$, $B$ and $\beta$, $C$ and $\gamma$ of the quadrilateral $Q$, the barycenter $\Gamma$ of the six vertices is the barycenter of the points $M$, $N$, and $P$ and, like these, it is on the Newton line $\delta$ of $Q$. Moreover, the center of gravity $G_a$ of the area of the triangle $A\beta\gamma$ is also the barycenter of the vertices $A$, $\beta$, $\gamma$, of this triangle; then, if $A''$ is the barycenter of the points $\alpha$, $B$, $C$, the point $\Gamma$ is the midpoint of $A''G_a$. Likewise, if $B''$ and $C''$ are barycenters of the points $\beta$, $C$, $A$ and $\gamma$, $A$, $B$, then $\Gamma$ is the midpoint of $B''G_b$ and $C''G_c$. It follows then that the triangle $T_g = A_gB_gC_g$ of the parallels to $BC$, $CA$, $AB$ through the points $G_a$, $G_b$, $G_c$ and the triangle $T$ are symmetrically equal and their center of symmetry $S_g$ coincides with the barycenter of the points $M$, $N$, $P$. Furthermore, the lines $\alpha A_g$, $\beta B_g$, $\gamma C_g$ are parallel to the line $\delta$ and are coincident with the lines $\alpha A_h$, $\beta B_h$, $\gamma C_h$ or $\delta_a$, $\delta_b$, $\delta_c$ of (b).

(e) If then we consider the points $D_a$, $D_b$, $D_c$ which cut the rectilinear segments $H_aG_a$, $H_bG_b$, $H_cG_c$ in the same ratio $k$, the vertices of the triangle $A_DB_DC_D$, obtained in drawing through the points $D_a$, $D_b$, $D_c$ the parallels to the sides $BC$, $CA$, $AB$, cut in the same ratio $k$ the dis-

tances $A_kA_\varepsilon$, $B_kB_\varepsilon$, $C_kC_\varepsilon$, the homologous vertices of triangles $A_kB_kC_k$ and $A_\varepsilon B_\varepsilon C_\varepsilon$; and when $k$ varies, the positions of triangle $A_DB_DC_D$ are those of triangle $A_1B_1C_1$ of (a) and (b) when the direction of the parallels $AA'$, $BB'$, $CC'$ varies.

(2) In particular, when $k=-3$, the points $D_a$, $D_b$, $D_c$ coincide with the points $O_a$, $O_b$, $O_c$ and the triangle $A_DB_DC_D$ with the triangle $T_0$ obtained by drawing through the points $O_a$, $O_b$, $O_c$ the parallels to the sides $BC$, $CA$, $AB$ of triangle $T$. It results then that triangle $T_0$ is symmetrically equal to triangle $T$ and that the center of symmetry of these triangles is situated on the Newton line $\delta$ of quadrilateral $Q$.*

Solved also by *Walter B. Clarke.*

No. 193. Proposed by *Walter B. Clarke,* San Jose, California.

Two unlimited straight lines intersect at $A$. Points $B$, $C$, $D$, and $E$ are taken anywhere on these two lines but $B$ and $D$ on one, $C$ and $E$ on the other. The line joining the mid-points of $BC$ and $DE$ intersects $AB$ at $G$ and $AC$ at $F$. The line joining the mid-points of $DE$ and $DC$ intersects $AB$ at $H$ and $AC$ at $J$. On $AC$ take $AK=AF$ and $AL=AJ$. Show that $HJ$ is parallel to $GK$ and that $HL$ is parallel to $GF$.

Solution by *C. A. Balof,* Lincoln College, Illinois.

Let $M$, $N$, $P$, and $Q$ be the mid-points of $BC$, $DE$, $BE$, and $DC$ respectively. Then $MQ$ and $PN$ are parallel to $AB$, and $MP$ and $QN$ are parallel to $AC$. Since angles $PNM$ and $AGF$ are equal, and angles $NMP$ and $GFA$ are equal, triangles $MPN$ and $AGF$ are similar. Thus $MP/AF$ equals $PN/AG$. But $AF$ and $PN$ are equal respectively to $AK$ and $MQ$. Hence $MP/AK$ equals $MQ/AG$. Since, also, angles $QMP$ and $GAK$ are equal, triangles $MPQ$ and $GAK$ are similar. But $GA$ and $AK$ are parallel to $MQ$ and $MP$ respectively. Therefore $GK$ is parallel to $QP$, i. e., to $HJ$.

Angles $MPQ$ and $QMP$ are equal respectively to angles $AJH$ and $HAJ$. Then triangles $MPQ$ and $AJH$ are similar, and $MQ/AH$ equals $MP/AJ$. Hence $PN/AH$ equals $MP/AL$. Also, angles $MPN$ and $LAB$ are equal. Thus triangles $MPN$ and $LAB$ are similar. But $MP$ and $PN$ are parallel respectively to $AL$ and $AH$. Therefore $MN$, i. e., $GF$, is parallel to $HL$.

*Questions proposed upon the same configuration appeared in Am. Math. Monthly, 1937, pp. 111, 395. See also an article by Thébault in Mathesis, 1937.

No. 208. Proposed by *Albert Farnell*, Louisiana State University, University, La.

Find the hodograph of a point on the ellipse which moves so that a line joining it with one of the foci covers equal areas in equal times.

Solution by the *Proposer*.

Using the equation of the ellipse with the origin at a focus:

$$r = p/(1 - e \cdot \cos \Theta);$$

the law of motion:

$$k = r^2\dot{\Theta}, \qquad (\cdot = d/dt)$$

we have $\qquad x = r \cdot \cos \Theta \quad \text{and} \quad y = r \cdot \sin \Theta.$

Thus $\qquad \dot{x} = -k \cdot \sin \Theta/p$

$$\dot{y} = k(\cos \Theta - e)/p.$$

Therefore the hodograph is a circle.*

No. 210. Proposed by *A. Moessner*, Nurnberg-N, Germany.

What is the general solution in integers of the system:

$$2(a^v + b^v) = x + y + z + w$$

$$2(a^{2v} + b^{2v}) = x^2 + y^2 + z^2 + w^2$$

$$2(a^{3v} + b^{3v}) = x^3 + y^3 + z^3 + w^3 \ ?$$

An example of such a solution is given by $v = 3$, $a = 2$, $b = 12$, $x = -336$, $y = 696$, $z = 1040$, $w = 2072$.

Solution by the *Proposer*.

The formula

$$2(m^2 + p^2)^2 = (m^2 - 2mp - p^2)^2 + (m^2 + 2mp - p^2)^2$$

provides solutions for the equation

$$(1) \qquad G^2 + G^2 = H^2 + K^2.$$

*See Ames and Murnaghan's Theoretical Mechanics, Prob. 27, p. 75.—Ed.

By a general result* in Number Theory, the truth of (1) implies the truth of

$$(2)\quad (T-G)^j+(T-G)^j+(T+G)^j+(T+G)^j$$
$$=(T-K)^j+(T-H)^j+(T+H)^j+(T+K)^j$$

for $j=1, 2, 3$. We thus obtain identities of the form

$$A, A, B, B \overset{3}{=} C, D, E, F.$$

Next choose $a$, $b$ and $v$ such that $b>a$ and that $2G$ is a divisor of each of the numbers $(b^v+a^v)$ and $(b^v-a^v)$. Then multiply (1) by $[(b^v-a^v)/2G]^2$ and obtain

$$\left(\frac{b^v-a^v}{2}\right)^2+\left(\frac{b^v-a^v}{2}\right)^2=\left[\frac{H(b^v-a^v)}{2G}\right]^2+\left[\frac{K(b^v-a^v)}{2G}\right]^2.$$

Now put $T=(b^v+a^v)/2$, and by (2) we shall have, since $T-(b^v-a^v)/2$ is $a^v$ and $T+(b^v-a^v)/2$ is $b^v$,

$$a^{vj}+a^{vj}+b^{vj}+b^{vj}=\left[T-K\left(\frac{b^v-a^v}{2G}\right)\right]^j+\left[T-H\left(\frac{b^v-a^v}{2G}\right)\right]^j$$
$$+\left[T+H\left(\frac{b^v-a^v}{2G}\right)\right]^j+\left[T+K\left(\frac{b^v-a^v}{2G}\right)\right]^j,$$

for $j=1, 2, 3$, which give the required solution.†

The cited example results from $5^2+5^2=1^2+7^2$ upon taking $a=10$, $b=60$, $v=3$; then $T=108500$ and we have

$$10^3, 10^3, 60^3, 60^3 \overset{3}{=} -42000, 87000, 130000, 259000,$$

which may now be simplified by suppression of the common factor $5^3$ (or even $10^3$).

*See Theorem 1 of Dorwart and Brown, *The Tarry-Escott Problem*, American Mathematical Monthly, Vol. XLIV (1937), p. 614. The notation, $a_1, a_2, \dots, a_s \overset{n}{=}, b_1, \dots, b_s$, means

$$\sum_{i=1}^{s} a_i^j = \sum_{i=1}^{s} b_i^j, \quad (j=1, 2, \dots, n).$$

The present equations are easily verified by direct expansion.—Ed.

†A remark by Chernick, *Ideal Solutions of the Tarry-Escott Problem*, American Mathematical Monthly, Vol. XLIV (1937), pp. 627-629 makes it appear that the above method is not completely general and that solutions exist which are not of this form.

No. 211. Proposed by *E. P. Starke,* Rutgers University.

Consider the series of Fibonacci (Leonardo of Pisa); 1, 1, 2, 3, 5, 8, 13, $\cdots$, where $a_{n+1}=a_{n-1}+a_n$, and show that, for every prime $p$,

(1) $a_p$ is relatively prime to all preceding terms, and

(2) there are infinitely many terms divisible by $p$.

(3) Also find necessary conditions on $r$ in order that $a_r$ shall be a prime.

Solution by the *Proposer.*

The following useful lemma has an easy proof:

*Let $r$ be the first subscript such that $a_r$ is divisible by a prime $p$; then the necessary and sufficient condition that $a_s$ be divisible by $p$ is that $s$ be a multiple of $r$.*

$a_{r+1}$ is not a multiple of $p$; for if it were, $a_{r-1}=a_{r+1}-a_r$ would be a multiple of $p$, contrary to the hypothesis on $r$. We have $a_{r+2}=a_{r+1}+a_r\equiv a_{r+1}=a_2a_{r+1}$ (mod $p$), $a_{r+3}=a_{r+2}+a_{r+1}\equiv 2a_{r+1}=a_3a_{r+1},\cdots,a_{r+k}\equiv a_ka_{r+1}$ (mod $p$). Thus, if $s=ur+v$, where $u$ and $v$ are integers and $0\leqq v<r$, we have

$$a_s\equiv a_{s-r}a_{r+1}\equiv a_{s-2r}a_{r+1}^2\equiv\cdots a_{v+r}a_{r+1}^{u-1}\equiv a_va_{r+1}^u.$$

Thus $a_s$ is congruent (mod $p$) to zero if and only if $v=0$.

The proof of (1) is obvious from the lemma. Lucas has proved* that the prime $p$ is a divisor of $a_{p+1}$ if $p$ is of the form $10d+3$, and a divisor of $a_{p-1}$ if $p=10d\pm1$. (5 divides $a_5$ and 2 divides $a_3$). With the lemma, this completes the proof of (2).

The lemma implies that $a_r$ is composite if $r$ is composite and $\neq 4$. To show this, let $t$ be a proper† factor of $r$ and consider $a_t$. Either $a_t$ is a prime $p$ or admits a prime $p$ as a proper divisor. Then $p$ is a proper divisor of $a_r$ and $a_r$ is composite. (If $t=2$ however, $a_t=1$ is neither prime nor composite—thus the result is untrue for $a_4$, since here $t=2$ is the *only* proper divisor of $r=4$.) But if $a_r$ is prime and $r>5$, by Lucas' result we must have $a_r$ a divisor of $a_{a_r\pm1}$, and hence our final condition: $r$ is a prime divisor of $a_r\pm1$, where the proper sign is chosen according to Lucas' result. That this condition is *not* also sufficient is seen by the example: $r=19$ is a divisor of $4181-1$, but $a_{19}=4181=37\cdot113$ is not prime.

*Abstract given in Dickson, *History of the Theory of Numbers,* v. i, p. 398.

†Thus $t$ satisfies $1<t<r$. We require further that $t>2$.

For further results see Archibald's paper in the American Mathematical Monthly (1918) pp. 235-238; also a paper by H. Gupta in Journal Indian Math. Soc. (1932) pp. 203-214. These references were supplied by *Morgan Ward*, who has done much work in connection with a generalized Fibonacci series which he calls Lucasian. See Bulletin Am. Math. Soc. (1934) pp. 825-828, also (1937) pp. 78-80 and a paper, *The Law of Apparition of Primes in a Lucasian Sequence*, in a current number of the Transactions Am. Math. Soc.

No. 212. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle having its Nagel point on one of its sides.

Solution by *W. T. Short*, Oklahoma Baptist University, Shawnee.

Let the Nagel point $N$ fall on side $b$ of the triangle $ABC$. It will then be symmetrical with respect to $A$ and $C$. Thus the reference system can be chosen so that points $A$, $B$, $C$, and $N$ will have coordinates: $(a,0)$, $(0,b)$, $(-a,0)$, and $(0,0)$, respectively. The equation of $AC$ is $bx+ay=ab$; of $BC$: $-bx+ay=ab$; of $AC$: $y=0$. The external bisector of angle $A$ is $bx+[a=\sqrt{(a^2+b^2)}]y=ab$, and the bisector of angle $B$ is $y=b$. These bisectors meet in $I'''$, the point whose coordinates are $[\sqrt{(a^2+b^2)}, b]$. Since $I'''N$ is perpendicular to $AB$ we have

$$-b/a=\sqrt{(a^2+b^2)}/b.$$

This gives, using the positive root:

$$b^2=a^2(1+\sqrt{2}).$$

Therefore construct the isosceles triangle with the base equal to $2a$ and the altitude equal to $\sqrt{(1+\sqrt{2})}$. The Nagel point will be at the mid-point of the base.

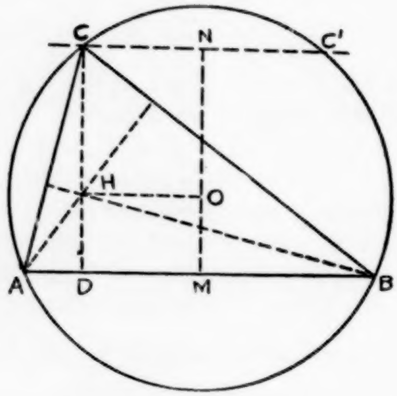
EDITOR'S NOTE: Unfortunately, there is considerable confusion abroad on the definition of the Nagel point. Johnson in *Modern Geometry*, p. 225, defines it as the intersection of the lines from the vertices of the triangle to the points of contact of the opposite escribed circles. Altshiller-Court in *College Geometry*, p. 105, defines it as the intersection of lines from the excenters perpendicular to the corresponding sides of the triangle. It is in this latter sense that the foregoing problem is solved. It should be noticed that the problem is then equivalent to the construction of a triangle $(I'I''I''')$ whose circumcenter lies on its pedal triangle.

No. 217. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle whose Euler line is parallel to a side. Under what conditions, expressed in terms of the sides, is this possible?

Solution by *C. W. Trigg*, Los Angeles City College.

The Euler line contains the orthocenter, circumcenter and centroid. The distance of the orthocenter from a vertex is equal to twice the distance of the circumcenter from the opposite side. So in circle $O$ with chord $AB$, drop a perpendicular from $O$ to $AB$ meeting $AB$ at $M$. On this perpendicular lay off $ON=2OM$ and on the opposite side of $O$ from $M$. Erect a perpendicular to $MN$ at $N$, cutting the circle at $C$ and $C'$. Then $ABC$ and $ABC'$ are triangles having their Euler lines parallel to $AB$. This follows since the orthocenter $H$ falls on the altitude $CD$ so that $CH=2OM$. But $CD=NM=3OM$ so $HD=OM$. Evidently $OM$ must be less than or equal to one-half the radius of the circle.




The necessary and sufficient conditions for the parallelism of the Euler line have been previously published* in terms of angles and of sides. That is, (a) the tangents of the angles form an arithmetic progression, whose mean is the tangent of the angle opposite the side to which the Euler line is parallel;

$$\text{(b)}(a^4+c^4-2b^4)+(a^2b^2+b^2c^2-2a^2c^2)=0,$$

where the Euler line is parallel to $b$. If this expression be solved for $b$, then

$$b=\tfrac{1}{2}\sqrt{(a^2+c^2)+\sqrt{(a^2+c^2)^2+8(a^2-c^2)^2}},$$

so that the triangle may also be constructed if $a$ and $c$ are given.

Also solved by *W. T. Short.*

*American Mathematical Monthly, E. 259, page 541, October, 1937. School Science and Mathematics, 1541, page 585, May, 1938.

No. 219. Proposed by *Jeannette Simpson*, student, New Jersey College for Women.

With ruler and compasses locate the point $P$ on the side $AB$ of any triangle $ABC$ such that the perpendicular from $P$ to $AC$ is the mean proportional between $AP$ and $PB$. Find also the point $Q$ such that the line $QE$ parallel to $BC$ and intercepted by $AC$ is the mean proportional between $AQ$ and $QB$.

Solution by *C. W. Trigg*, Los Angeles City College.

## PART II

On $CA$ as a base construct an isosceles triangle $CFA$ with legs equal to $a+c$. Lay off $CD=a=FG$, whence $DF=c=GA$. Draw $DA$ and $GC$ intersecting at $N$. Draw $FN$ and extend it to meet $CA$ at $E$. Through $E$ draw a parallel to $BC$ meeting $AB$ at $Q$, the required point.

Proof: In the triangle $CFA$, by Ceva's Theorem,

$$AE \cdot a^2 = EC \cdot c^2,$$

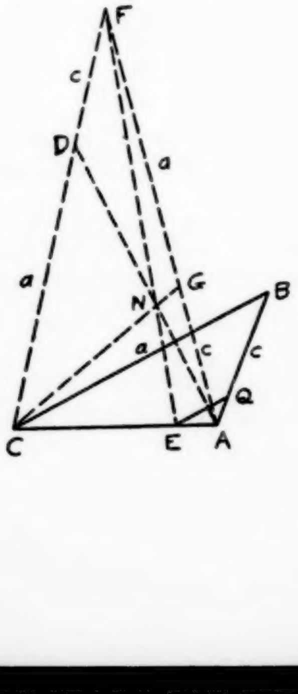
That is,

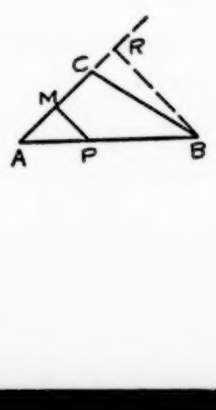
$$\frac{c^2}{a^2} = \frac{AE}{EC} = \frac{AQ}{QB} = \frac{AQ^2}{AQ \cdot QB}.$$

Now

$$\frac{c}{a} = \frac{AQ}{QE} \quad \text{so} \quad \frac{c^2}{a^2} = \frac{AQ^2}{QE^2}.$$

Therefore $QE^2 = AQ \cdot QB$.



## PART I

Draw $BR$ perpendicular to $AC$ or to $AC$ extended. To triangle $ARB$ apply the construction of Part I to locate $M$ on $AC$ and $P$ on $AB$. Then $MP$ is perpendicular to $AC$ and $MP^2 = AP \cdot PB$.

Also solved by *Yudell Luke.*

Late Solution: No. 207 by *C. W. Trigg.*

## PROPOSALS

No 15. Proposed by *W. E. Byrne,* Virginia Military Institute.

From Granville-Smith-Longley's Calculus, p. 113:

"What is the minimum value of

$$y = ae^{kx} + be^{-kx} \ ? \qquad \text{Ans. } 2\sqrt{ab}."$$

It happens that the answer is correct only under certain assumptions. What are these assumptions and what occurs in the other cases were $ab \neq 0$ ?

No. 33. Proposed by *F. M. Kenny,* Malone, N. Y.

The trees in an orchard of rectangular form are arranged so that the rows are the same distance apart as the trees in the rows. Find the arrangement if half of the trees are in the four rows bordering the rectangle.

No. 244. Proposed by *V. Thébault,* Le Mans, France.

Construct the squares $BCA_1A_2$, $CAB_1B_2$, $ABC_1C_2$ upon the sides of a triangle $ABC$. Show that the centers of gravity of the areas of the figures made up of the three squares, of the three squares and the triangle $ABC$, and of the hexagon $A_1A_2C_1C_2B_1B_2$ are collinear.

No. 245. Proposed by *V. Thébault,* Le Mans, France.

A sphere $S$ of given radius rolls upon a fixed sphere $S_1$.

(1) The center of similitude of $S$ and another fixed sphere $S_2$ describes a sphere.
(2) The radical plane of the spheres $S$ and $S_2$ envelopes a quadric.
(3) Examine the same questions if the sphere $S$ rolls upon two fixed spheres $S_1$ and $S_1'$.

No. 246. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

An old farmer had 30 children, 15 by his first wife, now deceased, and 15 by his second. This latter wife wished that her eldest son be heir to the property. Accordingly, she addressed her husband thus: "Let us settle this problem of who shall be your heir. We will arrange our 30 children in a circle, and remove every tenth child until there is left but one, who will succeed to your estate." The husband accepted the proposal, but as the process of elimination progressed he was bewildered to discover that the first 14 to be removed were his children by his first wife; and further, that the very next one to be eliminated would be her last child. In desperation he suggested that they count backwards beginning with this last child, in the hope that this child might be saved. The odds being apparently 15 to 1 in his wife's favor, she did not hesitate. Nevertheless, the farmer's son by his first wife became the heir. Required to know the arrangement of the children.*

No. 247. Proposed by *Walter B. Clarke*, San Jose, California.

Consider only triangles having circumcenter lying on a side of the orthic triangle.

(1) What is the largest possible angle in such a triangle?
(2) Construct one that is isosceles.

No. 248. Proposed by *M. S. Robertson*, Rutgers University.

For $f(x)=a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n$, we have $f(-x)=-f(x)$ if and only if $a_s$ vanishes for every even $s$. What is the corresponding functional equation for which the condition is $a_s=0$ for $s\equiv 0 \pmod{p}$, $p>2$?

No. 249. Proposed by *W. V. Parker*, Louisiana State University, University, La.

If the roots of $f(x)\equiv x^3+px+q=0$ are $\alpha+i\beta$, $\alpha-i\beta$, $-2\alpha$, where $\alpha$ and $\beta$ are real and different from 0, the roots of $f'(x)=0$ are the points in the complex plane midway between the center and foci of the ellipse with axes on the axes of coordinates and passing through $(\alpha,\beta)$, $(\alpha,-\beta)$ and $(-2\alpha, 0)$.

If the roots of $f(x)=0$ are $\alpha+\beta$, $\alpha-\beta$, $-2\alpha$, where $\alpha$ and $\beta$ are real and different from zero, the roots of $f'(x)=0$ are points in the

*See Hutton: Philosophical Recreations, p. 76.—Ed.

complex plane midway between the center and foci of the hyperbola with axes on the axes of coordinates and passing through $(\alpha, i\beta)$, $(\alpha, -i\beta)$ and $(-2\alpha, 0)$.

No. 250. Proposed by *Fred Fender*, New Brunswick, N. J.

Of all Pythagorean triangles, which is the one most nearly isosceles with sides less than 10,000 ?

---

From *The New York Sun*, September 6, 1938

## "Dark and Bloody Ground---"

(*The Freshman's Dream*)

In minuends of Algebra
  Wild corollaries twine;
The surds are full of variants,
  Of plus and minus sign;
Some whisper the binomial
  Quadratic makes his lair
Below the lush paranthesis,
  Whose roots are often square!

Among grim graphs of Algebra
  Odd exponenets some spy,
And logarithms formulate
  In terms of $x$ and $y$...
Remove the brackets, radicals,
  And do so with discretion,
Or you shall factor cubes, for terms
  Of infinite progression!

—*Harold Willard Gleason.*

# *Bibliography and Reviews*

*Edited by*
P. K. SMITH and H. A. SIMMONS

*An Introduction to Projective Geometry.* By C. W. O'Hara and D. R. Ward, Oxford at the Clarendon Press, 1937. ix+298 pages.

In their well written book, Fathers O'Hara and Ward do not claim that their text book has been written in order to "supply a long felt want." They hope that their work may do something to stimulate a demand for more widespread familiarity with projective geometry.

As the title implies, the aim of their book is to give the reader an elementary account of the fundamental concepts and methods of projective geometry of two dimensions, both synthetic and algebraic. Early in the Text, Euclidean prejudices are cleared away and an attempt is made "at substituting for them what may be called the projective mentality."

The first chapter gives a short historical introduction. In the next five, the synthetic method is developed. Starting with the initial propositions of incidence of lines and points one is lead up to a discussion of the more important properties of the conic. In the seventh and eighth chapters coordinate systems are introduced projectively, without an appeal to intuition. In the next two chapters the various metrical geometries, non-Euclidean and Euclidean, are discussed. After a short treatment of the theory of transformations, the book is brought to a close with a chapter on the study of the rôle of the special theory of relativity in the theory of physical geometry.

O'Hara and Ward's *Projective Geometry* represents a worth-while addition to the texts of the subject. The well constructed figures should prove helpful to the students in visualizing propositions. The well chosen assortment of problems is included in each chapter. The book appears to be a teachable text for either college seniors or graduate students.

*Agnes Scott College.* HENRY A. ROBINSON.

*Essentials of the Mathematics of Investment.* By Paul R. Rider, Farrar and Rinehart, Inc., New York, 1938. x+162 pages.

As indicated by the author this book aims to present the elements of the mathematics of investment and insurance in a clear but brief

form. Although a college course in algebra would be desirable as a preparatory course, two years of high school algebra should prove sufficient. Those topics from algebra which are needed are included in the first chapter, together with illustrative examples.

After a brief introduction to simple interest, the compound interest law is set up, using the interest conversion period as the unit of time. Unfortunately, in common with most writers on the mathematics of finance, this unit of time is replaced by the year in most of the formulas that appear later in the book. This chapter also considers nominal and effective rates of interest, present value, the principle of equivalence, discount, interest compounded continuously, and discount deducted continuously.

Annuities certain, perpetuities, and capitalized cost, each of which are completely discussed, appear in the next chapter. The author points out that the annuity formulas obtained by setting $m=p$ in the general formulas are essentially the same as the simple annuity formulas. Thus it would appear that formula (7), page 37, and formula (5), page 41, might well have been omitted.

The chapter on sinking funds includes the amortization and sinking fund methods of liquidating debts, valuation of mining property, and depreciation, including the unit cost method. A chapter on building and loan associations follows, but it is the opinion of the reviewer that for a brief book of this type it might well have been omitted and the extra space used in elaborating some of the essential topics.

The usual treatment of bonds is given, but the use of the formula for the premium in two different forms might prove somewhat confusing. It might have been better to have considered discount as a negative premium. The section on the purchase of bonds between interest payment dates is too brief. Under the quoted price with accrued interest plan, it would have been desirable to have given at least one method of finding the quoted price of the bond. Furthermore an unfortunate mistake on page 91 in the illustrative example and a poorly stated problem (3b, page 92) may cause a considerable amount of confusion.

As an introduction to life annuities and life insurance, a chapter on probability and the use of the mortality table is given. The treatment of both life annuities and life insurance is ample. The formulas are derived without the direct use of probability, which is probably easier for the average student. An obvious typographical error occurs in the last line of paragraph 62, page 108.

In general the problems are good, answers being given for a considerable number. The book is printed without a logarithm table but

includes the ordinary tables for the mathematics of finance, life annuities and life insurance.

On the whole this book is well written and will serve as an excellent text for the usual semester course of three hours per week. A mastery of the essentials as presented should well prepare the student for the more complex problems which often appear in business.

*University of California, Los Angeles.* CLIFFORD BELL.

*Advanced Analytic Geometry,* By Alan D. Campbell, John Wiley & Sons, New York, 1938. x+310 pages.

This book is separated into two parts dealing with affine plane analytic geometry and plane analytic projective geometry, respectively. In the first part the affine group is studied, complex elements are introduced and applications are made to algebraic point curves and linear families of conics. In the second part projective geometry is taken up after a discussion of the triangle of reference and homogeneous coordinates (point and line). Then applications are made to conics and algebraic curves. Frequent references are made to *Projective Geometry* by Veblen and Young.

The author uses consistently the name *alias* for a transformation considered as a change of axes and *alibi* for a transformaton acting upon points but leaving the reference frame unaltered. There is a large collection of problems with detailed hints to illustrate and complete the text. Many new concepts are introduced only after the convenience of more general treatment has been indicated by definite problems. Objection could be made to the use of the word *imaginary* instead of *complex*. On page 84 the point $(0,-1)$, not $(-1,0)$, is collinear with $(1,0)$ and $(2,1)$. On page 98 the first illustrative problem should read

$$y^2=2x^3+x^2 \quad \text{not} \quad y=2x^3+x^2$$

On page 169 the notation is confusing in: "we can change from nonhomogeneous to homogeneous coordinates by the equations

$$x=\frac{x}{z}, \quad y=\frac{y}{z},$$

remembering that the new $x$ and $y$ are not the same as the old $x$ and $y$." This text shows the influence of the classroom. It might well be used as an introduction to the more advanced treatise of Veblen and Young.

*Virginia Military Institute.* WILLIAM E. BYRNE.

*College Mathematics.* By M. A. Hill, Jr. and J. Burton Linker. Henry Holt and Company, 1938. xii+373+93 pages.

This text is a type of unified course in algebra, trigonometry, the elements of analytic geometry, and the elements of calculus. As stated in the preface, it is a text affording "an entrance into the broad field of mathematics and science", and suitable for "a sound basis for future work." The other class of students the authors hope may find the text useful are those "expecting to pursue non-scientific studies" who "wish to learn in a year's time something of the nature and scope of the subject".

The text is divided into two parts. Part I covers algebra and trigonometry. The authors exhibit psychological foresight in their weaving together the topics of algebra and trigonometry. The text begins with the elementary operations on polynomials, followed by the definition of the general angle and the definitions of the Cartesian and polar coordinates. The ground for the general definitions of the trigonometric functions is prepared by a definition of any function of a variable. Next, following a development of the reduction formulas, the solution of right triangles with the natural tables is treated. A study of algebraic factoring and fractions makes use of the trigonometric functions to aid the student in his grasp of the "knotty" trigonometric identities. A chapter on the theory of exponents is followed by logarithms. A chapter on linear equations in one or more unknowns is then followed by quadratic equations in one or more unknowns. In the chapter on quadratic equations the authors sensibly treat the solution of trigonometric equations. The theory of quadratic equations is followed by a chapter on the elements of the theory of equations, a chapter on functions of multiple angles and identities, and a chapter on the oblique triangle. Part I closes with a chapter on complex numbers.

Part II treats the elements of analytic geometry and the elements of differential and integral calculus. The usual topics of analytic geometry through the conic sections are treated. The analytics closes with a brief, but well-rounded, chapter on polar coordinates and parametric equations. The text closes with chapters XIX and XX on calculus. The differential calculus plunges immediately into several numerical examples of the derivative. Three fundamental theorems on limits then precede the formal definition of the derivative. The fundamental formulas on the differentiaton of the algebraic functions are followed by brief applications to the tangents and normals to a

curve, maxima and minima, and rates. The integral calculus goes no further into integration than the simple integral

$$\int u^n\, du.$$

The definite integral is treated and applied to the area of a curve and a few simple exercises in mechanics.

A weakness noted in the analytic geometry treated is in a lack of emphasis on graphing. There is a conspicuous absence in the treatment of graphs of algebraic functions, aside from the polynomials and conics.

The reviewer is of the opinion that the text is better suited to the first class of students mentioned in the beginning of the review. The material is ample for a five-hour course running for two semesters. The format is very pleasing, and for a department desiring a change in texts this book is highly worthy of examination.

*Louisiana Polytechnic Institute.* P. K. SMITH.

*Portraits of Eminent Mathematicians with Brief Biographical Sketches.* By David Eugene Smith, Professor Emeritus of Mathematics, Columbia University. Scripta Mathematica, New York, 1938. Portfolio II. Price $3.00.

This portfolio contains portraits and biographies of the following thirteen eminent mathematicians: Euclid, who lived about 300 B. C.; Cardan, 1501-76; John Kepler, 1571-1630; Pierre de Fermat, 1608-65; Blaise Pascal, 1623-62; Leonard Euler, 1703-83; Pierre-Simon, Marquis de Laplace, 1749-1827; Augustin-Louis Cauchy, 1789-1867; Carl Gustav Jacob Jacobi, 1804-51; William Rowan Hamilton, 1805-65; Arthur Cayley, 1821-95; Pafnutiy Lvovich Chebishef, 1821-94; Jules Henri Poincaré, 1854-1912.

The following are a few quotations taken from some of the biographies:

"Giorolamo Tiraboschi, historian of Italian literature, refers to Cardan as to a man who was more credulous of dreams than any frivolous women, . . . while at the same time he felt that he was one of the most profound and fertile genuises that Italy has ever produced, and that in mathematics and in medicine he had made rare and valuable discoveries."

"At an age when boys in our country would be in a high school class, he was a companion of world-renowned professors, opening for them new paths leading through

the vast maze of mathematics—still a puzzle to all who study his brief career and marvel at his achievements. Such a boy was Blaise Pascal."

"Laplace lived in a period of one of the world's greatest wars—the French Revolution. ... He was not always a careful writer, however, and often when he was himself puzzled he would write 'It is easy to see that...,' when it was not at all easy for even first-rate astronomers to see it at all."

"Life in the country village was, however, by no means easy, and the Cauchy family felt the pinch of poverty as did most of France. ... He had shown himself an indefatigable worker in all branches of mathematics— higher algebra, differential equations, the theory of probability, mathematics as applied to physics and astronomy, the calculus of variations, determinants, the foundations of mathematics, and the theory of functions. In all he published nearly eight hundred important memoirs distributed among these various branches."

"In the year 1900, however, mathematics had branched out so widely that no one could grasp the whole subject. It was in this period that Poincaré was in his prime, and of him it could have been said, without too much exaggeration, that he knew the greatest branches of the subject, particularly as applied to astronomy and physics, more completely than any of his contemporaries. As a matter of interest to teachers who place implicit confidence in the various types of tests it may be worth while to know that, when Poincaré had reached maturity, the Binet tests were tried upon him. It is said, on good authority, that the results showed him to be a man of extremely low intelligence—this man who in all the world ranked as one of the world's greatest scholars."

The biographies include accounts of the branches of mathematics in which the men worked and achieved success, the other fields of knowledge to which they contributed, and facts of human interest in their lives.

The portraits represent a variety of media including photographic reproductions of steel engravings, oil paintings, photographic portraits, and lithographs. The fact that the various portraits are done in the media which were most successfully used in the years in which the mathematicians lived would indicate that they are carefully selected reproductions. The portraits are appropriate for framing or display.

The portfolio would be a valuable addition to a mathematics library and useful as reference material in the study of the history of mathematics.

*Louisiana Polytechic Institute.* HENRY SCHROEDER.